1. Donada la successió definida recurrentment per $x_0 = \dfrac{1}{2}$ i $x_{n+1} = 3 \cdot 2^n \cdot x_n^2$, si $n \geq 0$, demostreu per inducció que $x_n = \dfrac{1}{6} \cdot 2^{-n} \cdot 3^{2^n}$, per a tot $n \geq 0$.

Resolució:

- Cas $n = 0$: $\quad x_0 = \dfrac{1}{6} \cdot 2^{-0} \cdot 3^{2^0} = \dfrac{1}{6} \cdot 1 \cdot 3^1 = \dfrac{1}{2}$.
- Suposem $x_n = \dfrac{1}{6} \cdot 2^{-n} \cdot 3^{2^n}$ i volem veure $x_{n+1} = \dfrac{1}{6} \cdot 2^{-(n+1)} \cdot 3^{2^{n+1}}$

  $$x_{n+1} = 3 \cdot 2^n \cdot x_n^2 = 3 \cdot 2^n \left[\frac{1}{6} \cdot 2^{-n} \cdot 3^{2^n}\right]^2 = 3 \cdot 2^n \cdot \frac{1}{6^2} 2^{-2n} \cdot 3^{2^{n+1}} = \frac{1}{6} \cdot 2^{-1} \cdot 2^{-n} \cdot 3^{2^{n+1}} = \frac{1}{6} \cdot 2^{-(n+1)} \cdot 3^{2^{n+1}}.$$

---

2. Sigui $h(x) = \dfrac{x^2+1}{x^2-1}$, definida per a $x > 1$, i $f(u,v) = \dfrac{v^2(u - v^2)}{u \cdot v^2 - 1}$. Calculeu $F(x) = f(h(x), h^{-1}(x))$ i trobeu el seu domini.

Resolució:

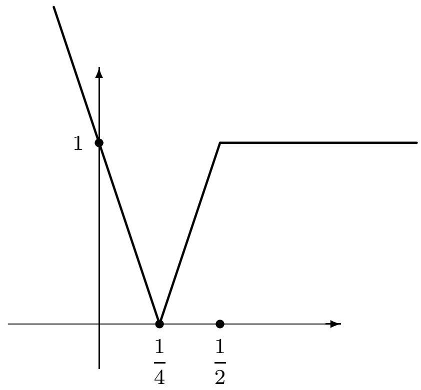
- $y = h(x) \Leftrightarrow y = \dfrac{x^2+1}{x^2-1} \Leftrightarrow y(x^2-1) = x^2+1 \Rightarrow x^2 = \dfrac{y+1}{y-1} \Leftrightarrow x = \sqrt{\dfrac{y+1}{y-1}}$(Observeu que volem $x > 1$).

  Així, $h^{-1}(x) = \sqrt{\dfrac{x+1}{x-1}}$ definida si $x > 1$.
- $h(x) \cdot (h^{-1}(x))^2 - 1 = \dfrac{x^2+1}{x^2-1} \cdot \dfrac{x+1}{x-1} - 1 = \dfrac{(x^2+1)}{(x-1)^2} - 1 = \dfrac{(x^2+1)-(x-1)^2}{(x-1)^2} = \dfrac{2x}{(x-1)^2} \neq 0 \;$ si $\; x > 1$.
- $[h^{-1}(x)]^2 (h(x) - (h^{-1}(x))^2) = \dfrac{x+1}{x-1}\left(\dfrac{x^2+1}{x^2-1} - \dfrac{x+1}{x-1}\right) = \dfrac{(x^2+1)-(x+1)^2}{(x-1)^2} = -\dfrac{2x}{(x-1)^2}$.
- $F(x) = \dfrac{-\dfrac{2x}{(x-1)^2}}{\dfrac{2x}{(2x-1)^2}} = -1$ ben definit si $x > 1$.

3. Identifiqueu el conjunt $A = \{x \in \mathbb{R} : \ |2x - |1-2x|| \leq 1\}$ i digueu si és obert o tancat.

Resolució: $f(x) = \big|2x - |1-2x|\big| = \left\{ \begin{array}{lll} |2x-(1-2x)| & \text{si} & x \leq \frac{1}{2} \\ |2x-(2x-1)| & \text{si} & x \geq \frac{1}{2} \end{array} \right\} = \left\{ \begin{array}{lll} |4x-1| & \text{si} & x \leq \frac{1}{2} \\ 1 & \text{si} & x \geq \frac{1}{2} \end{array} \right\} =$

$$= \left\{ \begin{array}{lll} 1-4x & \text{si} & x \leq \frac{1}{4} \\ 4x-1 & \text{si} & \frac{1}{4} \leq x \leq \frac{1}{2} \\ 1 & \text{si} & x \geq \frac{1}{2} \end{array} \right\}$$



$A = \{x \in \mathbb{R} : \ f(x) \leq 1\} = [0, +\infty)$ conjunt tancat!

---

4. Sigui $f(x) = \sqrt{x^2 - 5x} + x$. Trobeu el seu domini i les seves asímptotes $a \pm \infty$. Useu aquesta informació per fer un croquis aproximat de la seva gràfica. (No heu de calcular derivades de $f(x)$).

Resolució:

- $D_f = \{x \in \mathbb{R} : x^2 - 5x \geq 0\} = \{x \in \mathbb{R} : x(x-5) \geq 0\} = (-\infty, 0] \cup [5, +\infty]$
- A $+\infty$: $y = 2x - \frac{5}{2}$

$$m = \lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} \sqrt{1 - \frac{5}{x}} + 1 = 2$$
$$b = \lim_{x \to +\infty} (f(x) - mx) = \lim_{x \to +\infty} (\sqrt{x^2-5x} - x) = \lim_{x \to +\infty} \frac{-5x}{\sqrt{x^2-5x} + x} = -\frac{5}{2}$$

- A $-\infty$: $y = \frac{5}{2}$

$$m = \lim_{x \to -\infty} \frac{f(x)}{x} = \lim_{x \to -\infty} -\sqrt{1 - \frac{5}{x}} + 1 = 0$$
$$b = \lim_{x \to -\infty} (f(x) - mx) = \lim_{x \to -\infty} (\sqrt{x^2-5x} + x) = \lim_{x \to -\infty} \frac{-5x}{\sqrt{x^2-5x} - x} = \frac{5}{2}$$