

## 7. Derivació

1.

(a)  $f(x) = E(x) \sin x$  no és derivable a l'origen. El  $\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$  no existeix, ja que els límits laterals són diferents:

$$\lim_{h \rightarrow 0^+} \frac{E(h) \sin h}{h} = \lim_{h \rightarrow 0^+} \frac{0}{h} = 0, \quad \lim_{h \rightarrow 0^-} \frac{E(h) \sin h}{h} = \lim_{h \rightarrow 0^-} -\frac{\sin h}{h} = -1.$$

(b)  $f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{h^2 \chi_{\mathbb{Q}}(h) - 0}{h} = \lim_{h \rightarrow 0} h \chi_{\mathbb{Q}}(h) = 0.$

(c)  $f'(0) = \lim_{h \rightarrow 0} \frac{h|h| - 0}{h} = \lim_{h \rightarrow 0} |h| = 0.$

(d)  $f(x) = x \arctan \frac{1}{x}$  no és derivable a l'origen. El  $\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$  no existeix, ja que els límits laterals són diferents:

$$\lim_{h \rightarrow 0^+} \arctan \left( \frac{1}{h} \right) = \pi, \quad \lim_{h \rightarrow 0^-} \arctan \left( \frac{1}{h} \right) = -\pi.$$

2. Estudiem

$$f(x) = \begin{cases} \frac{\sqrt{x+1} - 1}{\sqrt[3]{x}}, & \text{si } x \neq 0, \\ 0, & \text{si } x = 0. \end{cases}$$

$f'(0) = \lim_{h \rightarrow 0} \frac{\frac{\sqrt{h+1} - 1}{\sqrt[3]{h}} - 0}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{h+1} - 1}{h \sqrt[3]{h}} = \lim_{h \rightarrow 0} \frac{1}{h^{1/3}(\sqrt{h+1} + 1)} = \infty.$  Llavors  $f$  no és derivable en  $x = 0$ .

3.  $\mathbf{R}$

Estudiem

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{si } x \neq 0, \\ 0, & \text{si } x = 0. \end{cases}$$

$f'(0) = \lim_{h \rightarrow 0} \frac{h^2 \sin \frac{1}{h} - 0}{h} = \lim_{h \rightarrow 0} h \sin \frac{1}{h} = 0.$  Llavors  $f$  és derivable en  $x = 0$ .

Funció derivada:

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x}, & \text{si } x \neq 0, \\ 0, & \text{si } x = 0. \end{cases}$$

$\lim_{x \rightarrow 0} f'(0)$  no existeix; per tant,  $f'(x)$  no és contínua en  $x = 0$ .

## 4. Sigui

$$\begin{aligned}
 f_1 &= |x| & f_1 &\in C^0(\mathbb{R}) \text{ però } f_1 \notin C^1(\mathbb{R}) \\
 f_2 &= \int_0^x f_1(x') dx' = \frac{x|x|}{2} \Rightarrow f_2 \in C^1(\mathbb{R}) \text{ però } f_2 \notin C^2(\mathbb{R}) \\
 f_3 &= \int_0^x f_2(x') dx' = \frac{x^2|x|}{3 \cdot 2} \Rightarrow f_3 \in C^2(\mathbb{R}) \text{ però } f_3 \notin C^3(\mathbb{R}) \\
 f_4 &= \int_0^x f_3(x') dx' = \frac{x^3|x|}{4!} \Rightarrow f_4 \in C^3(\mathbb{R}) \text{ però } f_4 \notin C^4(\mathbb{R}) \\
 f_5 &= \int_0^x f_4(x') dx' = \frac{x^4|x|}{5!} \Rightarrow f_5 \in C^4(\mathbb{R}) \text{ però } f_5 \notin C^5(\mathbb{R})
 \end{aligned}$$

Per tant, la funció  $f(x) = x^4|x|$  és tal que  $f \in C^4(\mathbb{R})$  i  $f \notin C^5(\mathbb{R})$ .

## 5. \*

$$6. \quad \frac{\partial f}{\partial x} = x^{y^x} \cdot y^x \cdot \left( \ln x \ln y + \frac{1}{x} \right); \quad \frac{\partial f}{\partial y} = x^{y^x} \cdot y^{x-1} \cdot x \ln x.$$

$$7. \quad \frac{\partial f}{\partial x} = 2x \tan \frac{y^2}{x^2 + y^2} - \frac{2x^3 y^2}{(x^2 + y^2)^2 \cos^2 \frac{y^2}{x^2 + y^2}}; \quad \frac{\partial f}{\partial y} = \frac{2x^4 y}{(x^2 + y^2)^2 \cos^2 \frac{y^2}{x^2 + y^2}}.$$

$$8. \quad D_y f(0, \pi) = \lim_{y \rightarrow \pi} \frac{f(0, y) - f(0, \pi)}{y - \pi} = \lim_{y \rightarrow \pi} \frac{-y^2 + \pi^2}{y - \pi} = \lim_{y \rightarrow \pi} -(y + \pi) = -2\pi.$$

$$9. \quad D_{xx} f = D_x(y^x \ln y) = y^x (\ln y)^2, \quad y > 0$$

$$D_{xy} f = D_y(y^x \ln y) = x y^{x-1} \ln y + y^{x-1}, \quad y > 0$$

$$D_{yx} f = D_x(x y^{x-1}) = y^{x-1} + x y^{x-1} \ln y, \quad y > 0$$

$$D_{yy} f = D_y(x y^{x-1}) = x(x-1) y^{x-2}, \quad y > 0$$

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$$11. \quad D_x^n f(x) = \sum_{k=0}^n \binom{n}{k} \frac{d^k(x^3)}{dx^k} \frac{d^{n-k} \cos x}{dx^{n-k}} = x^3 \frac{d^n \cos x}{dx^n} + 3n x^2 \frac{d^{n-1} \cos x}{dx^{n-1}} +$$

$$+ \binom{n}{2} 6x \frac{d^{n-2} \cos x}{dx^{n-2}} + \binom{n}{3} 6 \frac{d^{n-3} \cos x}{dx^{n-3}}, \quad \text{on } \frac{d^i \cos x}{dx^i} = \begin{cases} (-1)^{\frac{i}{2}} \cos x & \text{si } i \text{ és parell,} \\ (-1)^{\frac{i+1}{2}} \sin x & \text{si } i \text{ és senar.} \end{cases}$$

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$$14. \quad D_1 f(0, 0) = \lim_{x \rightarrow 0} \frac{f(x, 0) - f(0, 0)}{x - 0} = \lim_{x \rightarrow 0} -2 = -2.$$

$$D_2 f(0, 0) = \lim_{y \rightarrow 0} \frac{f(0, y) - f(0, 0)}{y - 0} = \lim_{y \rightarrow 0} \frac{0}{y} = 0.$$

$$15. \quad D_y f(x, 0) = \lim_{y \rightarrow 0} \frac{f(x, y) - f(x, 0)}{y - 0} = \lim_{y \rightarrow 0} y \sin \frac{x}{y} = 0.$$

$$D_y f(x, y) = 2y \sin \frac{x}{y} - x \cos \frac{x}{y}, \quad y \neq 0.$$

$$\lim_{y \rightarrow 0} D_y f(x, y) = 0 - x \lim_{y \rightarrow 0} \cos \frac{x}{y} \text{ no existeix, aleshores } D_y f(x, y) \text{ no és contínua.}$$

16. \*

$$(a) \quad f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases} \quad \text{No és contínua a l'origen:}$$

$$\lim_{x \rightarrow 0} f(x, x) = \lim_{x \rightarrow 0} \frac{x^2}{2x^2} = \frac{1}{2}, \quad \text{i} \quad \lim_{x \rightarrow 0} f(x, -x) = \lim_{x \rightarrow 0} \frac{-x^2}{2x^2} = -\frac{1}{2}.$$

$$D_x f(0, 0) = \lim_{x \rightarrow 0} \frac{f(x, 0) - f(0, 0)}{x - 0} = \lim_{x \rightarrow 0} \frac{0 - 0}{x} = 0; \quad D_y f(0, 0) = 0.$$

Les derivades parcials no poden ser contínues perquè si ho fossin,  $f$  seria  $C^1$ , però no és contínua.

$$(b) \quad g(x, y) = \begin{cases} \frac{x^2 y}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases} \quad \text{És contínua a l'origen:}$$

$$|g(r \cos \theta, r \sin \theta)| \leq \left| \frac{r^3}{r^2} \sin \theta \right| \leq r \rightarrow 0.$$

$$D_x g(0, 0) = 0 \quad D_x g(x, y) = \frac{2xy^3}{(x^2 + y^2)^2}; \quad D_y g(0, 0) = 0, \quad D_y g(x, y) = \frac{x^2(x^2 - y^2)}{(x^2 + y^2)^2}.$$

$$\lim_{(0,0)} D_x g(x, y) \text{ no existeix; } D_x g(x, \alpha x) \rightarrow \frac{2\alpha^3}{(1 + \alpha^2)^2}.$$

$$\lim_{(0,0)} D_y g(x, y) \text{ no existeix; } D_y g(x, \alpha x) \rightarrow \frac{1 - \alpha^2}{(1 + \alpha^2)^2}.$$

Les derivades parcials no són contínues.

$$(c) \quad h(x, y) = y \sin \frac{1}{x}, \quad x \neq 0; \quad h(0, y) = 0.$$

$$\lim_{x \rightarrow 0} h(x, y) = \lim_{x \rightarrow 0} y \sin \frac{1}{x} = y \lim_{x \rightarrow 0} \frac{1}{x} \text{ no existeix } \Rightarrow h \text{ no és contínua a } (0, y).$$

$$\frac{\partial h}{\partial x}(0, y) = \lim_{h \rightarrow 0} \frac{y \sin \frac{1}{h}}{h} \text{ no existeix.}$$

$$\frac{\partial h}{\partial y}(0, y) = \lim_{h \rightarrow 0} \frac{0 - 0}{h} = 0.$$

$$\frac{\partial h}{\partial y}(x, y) = \sin \frac{1}{x}, \quad x \neq 0.$$

$$\lim_{x \rightarrow 0} \frac{\partial h}{\partial y}(x, y) \text{ no existeix } \Rightarrow \frac{\partial h}{\partial y} \text{ no és contínua a } (0, y).$$

$$17. \quad f(x, y) = \begin{cases} \frac{x \sin y - y \sin x}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

$$D_x f(x, y) = \frac{\partial f}{\partial x}(x, y) = \frac{(\sin y - y \cos x)(x^2 + y^2) - 2x(x \sin y - y \sin x)}{(x^2 + y^2)^2} \quad \text{si } (x, y) \neq (0, 0)$$

$$D_x f(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0}{h^3} = 0.$$

$$D_{xx} f(0, 0) = \lim_{h \rightarrow 0} \frac{D_x f(h, 0) - D_x f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{0}{h^4} - 0}{h} = 0.$$

$$D_y f(x, y) = \frac{\partial f}{\partial y}(x, y) = \frac{(x \cos y - \sin x)(x^2 + y^2) - 2y(x \sin y - y \sin x)}{(x^2 + y^2)^2}, \quad \text{si } (x, y) \neq (0, 0).$$

$$D_y f(0, 0) = \lim_{h \rightarrow 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{0}{h^2} - 0}{h} = 0.$$

$$\begin{aligned} D_{yx} f(0, 0) &= \lim_{h \rightarrow 0} \frac{D_y f(h, 0) - D_y f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{(h - \sin h)h^2}{(h^2)^2} - 0}{h} = \lim_{h \rightarrow 0} \frac{h - \sin h}{h^3} = \lim_{h \rightarrow 0} \frac{1 - \cos h}{3h^2} = \\ &= \lim_{h \rightarrow 0} \frac{\sin h}{6h} = \lim_{h \rightarrow 0} \frac{\cos h}{6} = \frac{1}{6}. \end{aligned}$$

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$$(a) D_v f(a) = \nabla f(a) \cdot v; \quad \nabla f(x, y, z) = (2x + 2y, 2x, -3z^2); \quad \nabla f(a) = (22, 14, -3)$$

$$\Rightarrow D_v f(a) = (22, 14, -3) \frac{1}{\sqrt{14}} (1, 2, 3) = \frac{1}{\sqrt{14}} (22 + 28 - 9) = \frac{41}{\sqrt{14}}.$$

$$(b) v_{\max} = \frac{\nabla f(a)}{\|\nabla f(a)\|} = \frac{1}{\sqrt{22^2 + 14^2 + 3^2}} (22, 14, -3) = \frac{1}{\sqrt{689}} (22, 14, -3).$$

34. Sigui  $v = (v_1, v_2)$  tal que  $v_1^2 + v_2^2 = 1$ .

$$\begin{aligned} D_v f(0, 0) &= \lim_{t \rightarrow 0} \frac{f(tv_1, tv_2) - f(0, 0)}{t} = \lim_{t \rightarrow 0} \frac{t^{2/3} \sqrt[3]{v_1 v_2}}{t} = \\ &= \lim_{t \rightarrow 0} \frac{\sqrt{v_1 v_2}}{t^{1/3}} = \begin{cases} \text{no existeix si } v_1 \cdot v_2 \neq 0 \\ 0 \text{ si } v_1 \cdot v_2 = 0 \Leftrightarrow \begin{cases} v_1 = 0 \\ \text{o} \\ v_2 = 0. \end{cases} \end{cases} \end{aligned}$$

$$\begin{aligned} 35. D_{-v} f(a) &= \lim_{t \rightarrow 0} \frac{f(a + t(-v)) - f(a)}{t} = \lim_{s \rightarrow 0} \frac{f(a + sv) - f(a)}{-s} = \\ &= - \lim_{s \rightarrow 0} \frac{f(a + sv) - f(a)}{s} = -D_v f(a). \end{aligned}$$

$$36. D_{v_1} f(0, 2) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \Big|_{(0,2)} \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) = -\frac{1}{\sqrt{2}} \implies \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = -1 \quad (1)$$

$$D_{v_2} f(0, 2) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \Big|_{(0,2)} \left( -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) = \frac{3}{\sqrt{2}} \implies -\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 3 \quad (2)$$

$$(1) + (2) \implies 2 \frac{\partial f}{\partial y}(0, 2) = 2 \implies \frac{\partial f}{\partial y}(0, 2) = 1 \quad (1) \implies \frac{\partial f}{\partial x}(0, 2) = -2.$$

Per tant,  $\nabla f(0, 2) = (-2, 1)$ .

37.

$$\left. \begin{aligned} D_1(f \circ g) &= f'(g) \cdot D_1 g \\ D_2(f \circ g) &= f'(g) \cdot D_2 g \\ D_3(f \circ g) &= f'(g) \cdot D_3 g \end{aligned} \right\} \Rightarrow \|\nabla h\|^2 = [f'(g)]^2 ((D_1 g)^2 + (D_2 g)^2 + (D_3 g)^2) = \\ = [f'(g)]^2 (4(x^2 + y^2 + z^2)) = 4g(x, y, z)[f'(g)]^2.$$

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43. Tenim les equacions  $F_x \dot{x} + F_y \dot{y} = 0$ ,  $\dot{x}^2 + \dot{y}^2 = c$ . Derivant dues vegades la primera i una vegada la segona:

$$1) \quad \partial_{xx}^2 F(\dot{x})^2 + \partial_{yy}^2 F(\dot{y})^2 + 2\partial_{xy}^2 F \dot{x}\dot{y} + \partial_x F \ddot{x} + \partial_y F \ddot{y} = 0$$

$$2) \quad \dot{x}\ddot{x} + \dot{y}\ddot{y} = 0$$

Substituint  $\ddot{x}$  en la primera equació s'obté

$$\ddot{y} = \frac{-\dot{x}}{\dot{x}\partial_y F - \dot{y}\partial_x F} \left[ \partial_{xx}^2 F(\dot{x})^2 + \partial_{yy}^2 F(\dot{y})^2 + 2\partial_{xy}^2 F \dot{x}\dot{y} \right],$$

i finalment per trobar  $\ddot{x}$  usem l'equació 2)

En el nostre cas tenim que  $\partial_x F(1, 1) = 3$ ;  $\partial_y F(1, 1) = 2$

$\partial_{xx}^2 F(1, 1) = 2$ ;  $\partial_{yy}^2 F(1, 1) = 0$ ;  $\partial_{xy}^2 F(1, 1) = 3$ ;  $\dot{x} = 1$ ;  $\dot{y} = -3 \implies$

$$\ddot{y} = \frac{16}{11} \text{ i per tant, } \ddot{x} = -\frac{\dot{y}\ddot{y}}{\dot{y}} = \frac{48}{11}.$$