8. Teoremes d'aproximació

1. (a) $T(\ell, g) = 1.5391$, $|T(\ell, g) - T(\overline{\ell}, \overline{g})| \leq 0,084172$. (b) $|T(\ell, g) - T(\overline{\ell}, \overline{g})| \leq 0.0245$. És menor.

2. La seva part entera és 0, ja que es troba entre 0 i 1.

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9. **

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11. **

$$\forall \alpha \in \mathbb{R},\ \exists x_0 \in]a, b[\text{ tal que } \frac{f'(x_0)}{f(x_0)} = \alpha ?$$

$$\frac{f'(x)}{f(x)} - \alpha = 0 \Longleftrightarrow \frac{d}{dx}\left(\ln(f(x)e^{-\alpha x})\right) = 0 \Longleftrightarrow \frac{d}{dx}\left(f(x)e^{-\alpha x}\right) = 0\,.$$

Així, basta amb aplicar el teorema de Rolle a la funció $f(x)e^{-\alpha x}$ a $[a, b]$.

12. Sigui $h(x) = f(x) - g(x)$. Aplicant el teorema de Bolzano a h' i sabent que h' és decreixent, $\exists x_0$ tal que $h'(x_0) = 0$. Si h té més de 2 zeros,el teorema de Rolle diu que, com a mínim h' té més d'un zero. Però

$$\left.\begin{array}{r} \lim\limits_{x\to-\infty} h(x) = \lim\limits_{x\to+\infty} h(x) = -\infty \\ h(1/2) > 0 \end{array}\right\} \Longrightarrow h \text{ ha de tenir exactament 2 zeros.}$$

13. $T_3(x) = \pi(x-1) - \pi(x-1)^2 + (\pi - (\pi^3/6))(x-1)^3$.

14. $$\frac{x+60}{x^3 - 10x^2 + 31x - 30} = \frac{-(63/2)}{x-3} + \frac{(62/3)}{x-2} + \frac{(62/6)}{x-5} =$$

$$= \frac{(21/2)}{1-(x/3)} - \frac{(31/3)}{1-(x/2)} - \frac{(13/6)}{1-(x/5)}$$

$$T_n(x) = \frac{21}{2} - \frac{31}{3} - \frac{13}{6} + \cdots + \left(\frac{21}{2\cdot 3^n} - \frac{31}{3\cdot 2^n} - \frac{13}{65^n}\right)x^n\,.$$

15.

$$f'(x) = -\frac{2x}{1+x^4} = -2x\left(\sum_{k=0}^{n}(-1)^k(x^4)^k + O((x^4)^{n+1})\right) = 2\sum_{k=0}^{n}(-1)^{k+1}x^{4k+1} + O(x^{4n+5})$$

$$f(x) = f(0) + \int_0^x f'(t)dt = \frac{\pi}{4} + 2\sum_{k=0}^{n}(-1)^{k+1}\frac{x^{4k+2}}{4k+2} + O(x^{4n+6})\,.$$

16. $f(x) = \arcsin x \to f'(x) = \frac{1}{\sqrt{1-x^2}} = \sum_{k=0}^{n}\binom{-1/2}{k}(-x^2)^k + O((x^2)^{n+1}) \to$

$$f(x) = \sum_{k=0}^{n}\binom{-1/2}{k}(-1)^k\frac{x^{2k+1}}{2k+1} + O(x^{2n+3})$$

$$f(x) = \arctan x \to f'(x) = \frac{1}{1+x^2} = \sum_{k=0}^{n}(-1)^k(x^2)^k + O((x^2)^{n+1}) \to$$

$$f(x) = \sum_{k=0}^{n}(-1)^k\frac{x^{2k+1}}{2k+1} + O(x^{2n+3})$$

$$f(x) = \text{arcsinh }x \to \sinh f(x) = x \to f'(x)\cosh f(x) = 1 \to$$

$f'(x) = \frac{1}{\cosh f(x)} = \frac{1}{\sqrt{1+\sinh^2 f(x)}} = \frac{1}{\sqrt{1+x^2}}$, i fer anàleg al primer cas.

$$f(x) = \text{arctanh }x \to \tanh f(x) = x \to f'(x)\frac{1}{\cosh^2 f(x)} = 1 \to$$

$f'(x) = \cosh^2 f(x) = \frac{1}{1-\tanh^2 f(x)} = \frac{1}{1-x^2}$, i fer anàleg al segon cas.

17.

(a) $\cos(1-cosx) \simeq 1 - \frac{(1-\cos x)^2}{2} \simeq 1 - \frac{\left(\frac{x^2}{2}\right)^2}{2} = 1 - \frac{x^4}{8} \Rightarrow$ el límit demanat és $\frac{1}{8}$.

(b) $\ln\cos\sqrt{\frac{2a}{8}} \simeq \cos\sqrt{\frac{2a}{x}} - 1 \simeq -\frac{1}{2}\left(\sqrt{\frac{2a}{x}}\right)^2 = -\frac{a}{x} \Rightarrow$ el límit demanat és e^{-a}.

18. (a) 0. (b) 1. (c) 0. (d) $\nexists$.

19.

(a) $\frac{x^2\sin(1/x)}{\sin x} = \frac{x}{\sin x}(x\sin(1/x)) \to 1\cdot 0 = 0$ i $\nexists \lim_0 \frac{2x\sin(1/x)-\cos(1/x)}{\cos x}$.

(b) canvi $x = \frac{1}{y}$; $\frac{x-\sin x}{x+\sin x} = \frac{(1/y)-\sin(1/y)}{(1/y)+\sin(1/y)} = \frac{1-y\sin(1/y)}{1+y\sin(1/y)} \underset{y\to 0}{\longrightarrow} 1$ i $\nexists \lim_\infty \frac{1-\cos x}{1+\cos x}$.

20. (a) 4. (b) $-1/2$.

21. Canvi $y = \frac{1}{x}$; $\frac{x^\beta-(x-1)^\beta}{x^\beta}x = \frac{1-(1-y)^\beta}{y} = \frac{1-(1-\beta y+O(y^2))}{y} = \beta + O(y) \underset{y\to 0}{\longrightarrow} \beta$.

22 .

i) $f(x) = 2x + \frac{4}{3}x^3 + \frac{23}{60}x^5 + O(x^7)$.

ii) $\frac{1}{5!}D^5f(0) = \frac{23}{60} \to D^5f(0) = 46$. Busquem $f^{-1}(y) = A_0 + A_1y + A_2y^2 + A_3y^3 + O(y^4)$ tal que $f^{-1}(f(x)) = x \to A_3 = -\frac{1}{12}$; $\frac{1}{3!}D^3f^{-1}(0) = -\frac{1}{12} \to D^3f^{-1}(0) = -\frac{1}{2}$.

23 . $f(x) = x - \frac{x^2}{6} + O(x^4)$. Per trobar f^{-1} fem el mateix que a (ii) del problema **22**: $A_3 = \frac{1}{18} \to D^3f^{-1}(0) = \frac{1}{3}$.

24 . $\sin x = x - \frac{x^3}{6} + R_5$ amb $|R_5| \le \frac{1}{5!}|x|^5$; $x - \frac{x^3}{6} = x^2 \to$ una solució és $x_0 = \sqrt{15} - 3$; $|\sin x_0 - x_0^2| = |R_5| \le \frac{1}{120}x_0^5 < 0.005$.

25 . *

i) $e^3 = \lim\limits_{x\to 0}\left(1 + x + \frac{f(x)}{x}\right)^{1/x} = e^{\lim\limits_{x\to 0}\frac{1}{x}\ln\left(1+x+\frac{f(x)}{x}\right)} \Longrightarrow$

$$\Longrightarrow \lim_{x\to 0}\left(1 + x + \frac{f(x)}{x}\right) = 1 \Longrightarrow f'(0) = \lim_{x\to 0}\frac{f(x)}{x} = 0 \Longrightarrow f(0) = 0.$$

ii) Sabent que tots els límits implicats existeixen, apliquem l'Hôpital

$$3 = \lim_{x\to 0}\frac{1}{x}\ln\left(1 + x + \frac{f(x)}{x}\right) = \lim_{x\to 0}\frac{1 + f''(x) - \frac{f'(x)}{x} + \frac{f(x)}{x^2}}{1 + 2x + f'(x)} =$$

$$= \frac{1 + f''(0) - f''(0) + \frac{1}{2}f''(0)}{1 + f'(0)} \Longrightarrow f''(0) = 4.$$

26 .

(a) $F(x) = f(x) - x$ és de classe C^2 (diferència de funcions de classe C^2);

$F(0) = f(0) - 0 = 0$,

$F(1) = f(1) - 1 = 1 - 1 = 0$.

Aplicant el lema de Rolle $\exists\, \alpha \in (0,1)$ tal que $F'(\alpha) = f'(\alpha) - 1 = 0$, aleshores $f'(\alpha) = 1$.

(b) Aplicant el teorema del valor mig a f': $f'(\alpha) - f'(0) = f''(\eta)(\alpha - 0)$ amb $\eta \in (0,\alpha)$. Aleshores, $f''(\eta) = \frac{f'(\alpha) - f'(0)}{\alpha} = \frac{1 - f'(0)}{\alpha} < 0$, ja que $f'(0) > 1$ i $\alpha \in (0,1)$ (i per tant és $\alpha > 0$). Tenim llavors que $f''(0) > 0$ i $f''(\eta) < 0$ amb $\eta \in (0,\alpha) \subset (0,1)$. Com que f és de classe C^2, f'' és una funció contínua i aplicant el teorema de Bolzano a l'interval $[0,\eta]$, tenim que $\exists\, 0 < \beta < \eta$ $(< \alpha < 1)$, tal que $f''(\beta) = 0$.

27 . *

$\dfrac{x+ax^3}{1+bx} = (x+ax^3)(1-bx^2+b^2x^4-b^3x^6+\cdots) = x-bx^3+b^2x^5-b^3x^7+\cdots+ax^3-abx^5+ ab^2x^7-\cdots = x+(a-b)x^3-b(a-b)x^5+b^2(a-b)x^7+O(x^7)$.

$$\lim_{x\to 0}\frac{\sin x-\dfrac{x+a^3}{a+bx^2}}{x^p} = \lim_{x\to 0}\frac{-\left(a-b+\frac{1}{3!}\right)x^3+\left[b(a-b)+\frac{1}{5!}\right]x^5-\left[b^2(a-b)+\frac{1}{7!}\right]x^7+O(x^7)}{x^p}$$

Si volem $p\in\mathbb{N}$ el més gran possible caldrà agafar a,b tals que:

$$\left.\begin{aligned} a-b&=-\frac{1}{3!}\\ b(a-b)&=-\frac{1}{5!}\end{aligned}\right\}\to -\frac{b}{3!}=-\frac{1}{5!}\to b=\frac{3!}{5!}=\frac{1}{20},\quad a=b-\frac{1}{3!}=\frac{1}{20}-\frac{1}{6}=-\frac{7}{60}\,;$$

llavors, $b^2(a-b)+\dfrac{1}{7!}=\dfrac{1}{400}\left(-\dfrac{1}{3!}\right)+\dfrac{1}{7!}=-\dfrac{1}{2400}+\dfrac{1}{5040}=-\dfrac{11}{50400}\neq 0$; aleshores:

$$\lim_{x\to 0}\frac{\sin x-\dfrac{x+a^3}{a+bx^2}}{x^p}=\lim_{x\to 0}\frac{\dfrac{11}{50400}x^7+O(x^7)}{x^p}=0\quad\text{amb } p\in\mathbb{N}\Rightarrow p=6\,.$$

28 . *

$$\sqrt{x^2-x+1}=x\left[1-\left(\frac{1}{x}-\frac{1}{x^2}\right)\right]^{1/2}=x\left[1-\frac{1}{2}\left(\frac{1}{x}-\frac{1}{x^2}\right)+0\left(\frac{1}{x^2}\right)\right]$$

$$\sqrt[3]{x^3+\lambda x^2+1}=x\left[1-\left(\frac{1}{x^3}+\frac{\lambda}{x}\right)\right]^{1/3}=x\left[1+\frac{1}{3}\left(\frac{\lambda}{x}+\frac{1}{x^3}\right)+0\left(\frac{1}{x^3}\right)\right]$$

$$\sqrt{x^2-x+1}-\sqrt[3]{x^3+\lambda x^2+1}=x\left(\left[1-\frac{1}{2x}+0\left(\frac{1}{x}\right)\right]-\left[1+\frac{\lambda}{3x}+0\left(\frac{1}{2}\right)\right]\right)=$$

$$=-x\left(\frac{1}{2}+\frac{\lambda}{3}\right)\frac{1}{x}+0\left(\frac{1}{x}\right)=-\left(\frac{1}{2}+\frac{\lambda}{3}\right)+0\left(\frac{1}{x}\right)\underset{x\to+\infty}{\longrightarrow}0\iff\lambda=-\frac{3}{2}.$$

29 . *

Desenvolupant f per Taylor fins a 1r ordre en $x=a$ i escrivint el terme complementari de Lagrange:

$$f(x)=f(a)+f'(a)(x-a)+\frac{1}{2!}f''(\xi)(x-a)^2\,,\qquad (*)$$

amb ξ entre x i a, i.e., $|\xi-a|<|x-a|$,

$$(*)\iff f'(a)(x-a)=f(x)-f(a)-\frac{1}{2!}f''(\xi)(x-a)^2\,,$$

d'on prenent valors absoluts i aplicant la desigualtat triangular,

$$\begin{aligned}|f'(a)|\cdot|x-a| &= |f(x)-f(a)-\frac{1}{2!}f''(\xi)(x-a)^2|\\ &\le |f(x)|+|f(a)|+\frac{1}{2}|f''(\xi)|\cdot|x-a|^2\le 2C_0+\frac{1}{2}C_2|x-a|^2\,.\end{aligned}$$

En particular, per a $x=a+2\sqrt{\dfrac{C_0}{C_2}}$ la desigualtat de dalt queda

$$2\sqrt{\frac{C_0}{C_2}}|f'(a)|\le 4C_0\,,$$

i.e.,

$$|f'(a)| \leq 2C_0\sqrt{\frac{C_2}{C_0}} = 2\sqrt{C_0C_2}\,,$$

que és el que es volia provar.

30 .

(a) $f(x,y) = (x-1+1)^3 + (y-2+2)^3 + (x-1+1)(y-2+2)^2 =$

$$= 13 + 7(x-1) + 16(y-2) + 3(x-1)^2 + 4(x-1)(y-2) + 7(y-2)^2 +$$
$$+ (x-a)^3 + (x-1)(y-2)^2 + (y-2)^3 + 0(\|(x-1,\, y-2)\|^3)$$

(b) $g(x,y) = \ln(x+y) = \ln 2 + \ln\left(1 + \dfrac{x-1}{2} + \dfrac{y-1}{2}\right) =$

$$= \left\{ \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} \right\} =$$

$$= \ln 2 + \frac{x-1}{2} + \frac{y-1}{2} - \frac{1}{2}\left(\frac{x-1}{2} + \frac{y-1}{2}\right)^2 + \frac{1}{3}\left(\frac{x-1}{2} + \frac{y-1}{2}\right)^3 + 0(\|(x-1,\, y-1)\|^3) =$$

$$= \ln 2 + \frac{1}{2}(x-1) + \frac{1}{2}(y-1) - \frac{1}{8}(x-1)^2 - \frac{1}{4}(x-1)(y-1) - \frac{1}{8}(y-1)^2 +$$
$$+ \frac{1}{24}(x-1)^3 + \frac{1}{8}(x-1)^2(y-1) + \frac{1}{8}(x-1)(y-1)^2 + \frac{1}{24}(y-1)^3 + 0(\|(x-1,\, y-1)\|^3)$$

(c) $h(x,y,z) = e^{a(x+y+z)} = 1 + a(x+y+z) + \dfrac{a^2}{2}(x+y+z)^2 + \dfrac{a^3}{6}(x+y+z)^3 + 0(\|(x,y,z)\|^3) =$

$$= 1 + ax + ay + az + \frac{a^2}{2}x^2 + a^2xy + a^2xz + \frac{a^2}{z}y^2 + a^2yz + \frac{a^2}{2}z^2 + \frac{a^3}{6}x^3 + \frac{a^3}{2}x^2y +$$
$$+ \frac{a^3}{2}x^2z + \frac{a^3}{6}y^3 + \frac{a^3}{2}y^2x + \frac{a^3}{2}y^2z + \frac{a^3}{6}z^3 + \frac{a^3}{2}z^2x + \frac{a^3}{2}z^2y + a^3xyz + 0(\|(x,y,z)\|^3)$$

31 . $f(x,y) = x^y = y^{y\ln x} = e^{(y-1)\ln(1+(x+1)) + \ln(1+(x-1))} =$

$$= (1 + (x-1))\exp\left((y-1)\left((x-1) - \frac{(x-1)^2}{2} + 0((x-1)^3)\right)\right) =$$

$$= (1 + (x-1))\exp\left((y-1)(x-1) - \frac{1}{2}(x-1)^2(y-1) + 0(\|(x-1,\, y-1)\|)^3\right) =$$

$$= (1 + (x-1))\left(1 + (x-1)(y-1) - \frac{1}{2}(x-1)^2(y-1) + 0(\|(x-1,\, y-1)\|^3)\right) =$$

$$= 1 + (x-1) + (y-1)(x-1) + \frac{1}{2}(x-1)^2(y-1) + 0(\|(x-1,\, y-1)\|^3)$$

$$(1'1)^{1'02} = f(1'1, 1'02) \approx 1 + 0'1 + 0'1 \cdot 0'02 + \frac{1}{2}0'1^2 \cdot 0'02 \approx 1'1021\ldots$$

Un càlcul més precís (amb 12 xifres decimals correctes) produeix:

$$1'1^{1'02} = 1'102\ 098\ 823\ 713\,.$$

32 . *

D'una banda,

$$\begin{aligned} e^x \sin(x+y) &= \left(1+x+\frac{x^2}{2}+\cdots\right)\left((x+y)-\frac{(x+y)^3}{3!}+\cdots\right) = \\ &= x+y+x(x+y)+0(\|(x,y)\|)^3 = x+y+x^2+xy+0(\|(x,y)\|^2)\,; \end{aligned}$$

$e^x \sin(x+y) \in C^\infty(\mathbb{R}^2)$ (desenvolupament de Taylor per generació). D'aquí;

$$\lim_{(0,0)} \frac{e^x\sin(x+y)-x-y-x^2-xy}{(|x|+|y|)^2} = \lim_{(0,0)} \frac{0(\|(x,y)\|^2)}{\|(x,y)\|^2}\cdot\frac{\|(x,y)\|^2}{(|x|+|y|)^2} = 0\,,$$

ja que $0 < \dfrac{\|(x,y)\|^2}{(|x|+|y|)^2} = \dfrac{x^2+y^2}{x^2+y^2+2|xy|} < 1\,, \quad \forall\,(x,y)\neq(0,0)\,.$

33 . *

$$\begin{aligned} \ln(x-1+e^y) &= \ln(1+(x-1)+e^y-1) = \ln(1+(x-1)+y+0(\|(x-1,\,y)\|)) = \\ &= (x-1)+y+0(\|(x-1,\,y)\|) \end{aligned}$$

(Taylor per generació), d'on

$$\lim_{(0,0)} \frac{\ln(x-1+e^y)-\lambda(x+y-1)}{\sqrt{(x-1)^2+y^2}} = \lim_{(1,0)} \left[\frac{(1-\lambda)(x+y-1)}{\sqrt{(x-1)^2+y^2}} + \frac{0(\|(x-1,\,y\|))}{\|(x-1,\,y)\|}\right]\,.$$

Aquest límit clarament val 0 quan $\lambda = 1$. Si $\lambda\neq 1$, considerem el límit direccional segons la recta $x=1$.

$$\lim_{(1,0),\{x=1\}} \frac{\ln(x-1+e^y)-\lambda(x+y-1)}{\sqrt{(x-1)^2+y^2}} = \lim_{y\to 0}(1-\lambda)\frac{y}{|y|} = \begin{cases} 1-\lambda, & \text{si } y\to 0^+,\\ -(1-\lambda), & \text{si } y\to 0^-.\end{cases}\,.$$

Llavors, aquest límit direccional (i en conseqüència el límit què estem estudiant) $\nexists$ quan $\lambda\neq 1$. Resumint:

$$\lim_{(1,0)} \frac{\ln(x-1+e^y)-\lambda(x+y-1)}{\sqrt{(x-1)^2+y^2}} = 0 \Longleftrightarrow \lambda = 1 \quad (\text{i } \nexists \text{ quan } \lambda\neq 1)\,.$$

34 . $\arctan(x^2+y)\in C^\infty(\mathbb{R}^2)$, i fent servir els criteris de generació:

$$\begin{aligned} \arctan(x^2+y) &= \arctan\circ(x^2+y) = (\text{exercici } 8.17) = \\ &= x^2+y-\frac{(x^2+y)^3}{3}+0((x^2+y)^3) = y+x^2-\frac{1}{3}y^3+0(\|(x,y)\|^3)\,. \end{aligned}$$

Aleshores,

$$\lim_{(0,0)} \frac{\arctan(x^2+y)-x^2-y+\lambda y^3}{\sqrt{(x^2+y^2)^3}} = \left(\frac{0(\|(x,y)\|^3)}{\|(x,y)\|^3} + \frac{\left(\lambda-\frac{1}{3}\right)y^3}{\|(x,y)\|^3}\right)\,.$$

I llavors, clarament el límit és $=0$ si $\lambda = \dfrac{1}{3}$. Si $\lambda\neq\dfrac{1}{3}$, considerant el límit segons l'eix vertical (i.e., segons $x=0$):

$$\begin{aligned} \lim_{(0,0),\{x=0\}} \frac{\arctan(x^2+y)-x^2-y+\lambda y^3}{\sqrt{(x^2+y^2)^3}} &= \lim_{y\to 0}\frac{\arctan y - y+\lambda y^3}{|y|^3} = \\ &= \lim_{y\to 0}\left[\left(\lambda-\frac{1}{3}\right)\frac{y^3}{|y|^3}+\frac{0(y^3)}{|y|^3}\right] = \begin{cases} \lambda-\dfrac{1}{3}, & \text{si } y\to 0^+,\\ -\left(\lambda-\dfrac{1}{3}\right) & \text{si } y\to 0^-.\end{cases} \end{aligned}$$

Per tant, aquest límit $\nexists$ si $\lambda \neq \frac{1}{3}$; i llavors $\nexists$ el límit excepte quan $\lambda = \frac{1}{3}$ i en aquest cas val 0, i.e.:

$$\lim_{(0,0)} \frac{\arctan(x^2+y) - x^2 - y + \lambda y^3}{\sqrt{(x^2+y^2)^3}} = 0 \Longleftrightarrow \lambda = \frac{1}{3}$$

i el límit $\nexists$ si $\lambda \neq \frac{1}{3}$.

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