Càlcul Infinitesimal/Càlcul I **Codi: 21702/21762** **A**

Temps: 50 minuts **Test** (8 punts) 19 de Juny de 2008

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1. Sigui $f : B_1(0,0) \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ definida per $f(x,y) = \dfrac{x^n}{\sin(x^2+y^2)}$ si $(x,y) \neq (0,0)$ i $f(0,0) = 0$. Digueu per a quins valors de $n \in \mathbb{N}$ és f contínua en $(0,0)$.

Resolució: Si calculem el límit en el $(0,0)$ sobre la recta $y = 0$ tenim:

$$\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{x^n}{\sin(x^2)} = \lim_{x\to 0} \frac{x^n}{x^2} = \lim_{x\to 0} x^{n-2} = \begin{cases} \text{divergent} & \text{si} \quad n = 1 \\ 1 & \text{si} \quad n = 2 \\ 0 & \text{si} \quad n > 2 \end{cases}$$

Així f només pot ser contínua en $(0,0)$ si $n > 2$. Usant polars:

$$|f(r\cos\theta,\, r\sin\theta) - 0| = \left| \frac{r^n(\cos\theta)^n}{\sin(r^2\cos^2\theta + r^2\sin^2\theta)} \right| \leq \frac{r^n}{|\sin(r^2)|} \underset{r\to 0^+}{\longrightarrow} 0\,,$$

si $n > 2 \Longrightarrow \lim\limits_{(x,y)\to(0,0)} f = 0$ si $n > 2 \Longrightarrow f$ contínua en $(0,0)$ si $n > 2$.

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2. Demostreu que les equacions següents defineixen u i v com a funcions implícites de x i y en un entorn de $(x,y,u,v) = (1,1,1,1)$ i calculeu les derivades $D_x u(1,1)$ i $D_x v(1,1)$:

$$u^2 + v^2 - x^2 - y^2 = 0\,, \quad x \cdot v - y \cdot u = 0\,.$$

Resolució: $f(x,y,u,v) = u^2 + v^2 - x^2 - y^2$, $g(x,y,u,v) = x\cdot v - y \cdot u$.

(i) $f, g \in C^\infty$.

(ii) $f(1,1,1,1) = g(1,1,1,1) = 0$.

(iii) $\dfrac{\partial(f,g)}{\partial(u,v)} = \begin{pmatrix} 2u & 2v \\ -y & x \end{pmatrix}$; $\det \dfrac{\partial(f,g)}{\partial(u,v)}(1,1,1,1) = \begin{vmatrix} 2 & 2 \\ -1 & 1 \end{vmatrix} = 4 \neq 0$.

Podem aïllar $u = u(x,y)$ i $v = v(x,y)$, solució de les equacions, amb $u(1,1) = 1$ i $v(1,1) = 1$. Derivant les equacions respecte x obtenim:

$$\left.\begin{aligned} 2uD_x u + 2vD_x v - 2x = 0 \\ v + xD_x v - yD_x u = 0 \end{aligned}\right\}$$

En $x = y = 1$ i $u(1,1) = v(1,1) = 1$ tenim:

$$\left.\begin{aligned} 2D_x u(1,1) + 2D_x v(1,1) - 2 = 0 \\ 1 + D_x v(1,1) - D_x u(1,1) = 0 \end{aligned}\right\} \Longrightarrow D_x u(1,1) = 1, \quad D_x v(1,1) = 0\,.$$

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3. Sigui $p(x) = x^2 + x + 1$. Calculeu $\lim\limits_n \left[\dfrac{p(n+1)}{p(n)}\right]^n$.

Resolució:

$$\lim_n \left[\frac{(n+1)^2 + n + 1 + 1}{n^2+n+1}\right]^n = \lim_n \left[\frac{n^2+3n+3}{n^2+n+1}\right]^n = \lim_n \left[1 + \frac{2n+2}{n^2+n+1}\right]^n =$$

$$= \lim_n \left[1 + \frac{1}{\frac{n^2+n+1}{2n+2}}\right]^{\frac{n^2+n+1}{2n+2}\cdot\frac{(2n+2)n}{n^2+n+1}} = e^2.$$

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4. Donades les funcions $f(x,y) = (x + \ln y,\ y)$ i $g(x,y) = (x,\ y + e^x)$, calculeu $D(f^{-1}\circ g)(0,0)$.

Resolució: $g(0,0) = (0,1)$ i $f(0,1) = (0,1) \Longrightarrow f^{-1}(0,1) = (0,1)$.

$$\begin{aligned} D(f^{-1}\circ g)(0,0) &= Df^{-1}(g(0,0)) \cdot Dg(0,0) = [Df(f^{-1}(0,1))]^{-1} \cdot Dg(0,0) = \\ &= [Df(0,1)]^{-1} \cdot Dg(0,0) = \begin{pmatrix} 1 & 1/y \\ 0 & 1 \end{pmatrix}^{-1}_{|(0,1)} \cdot \begin{pmatrix} 1 & 0 \\ e^x & 1 \end{pmatrix}_{|(0,0)} = \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \end{aligned}$$
