Constructing Stein surfaces by topological isotopy

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Stein surfaces (affine complex analytic surfaces) can be studied using the methods of topological 4-manifold theory. It is well known that Stein surfaces admit open handle decompositions with all indices at most 2. In fact, this condition is also sufficient to guarantee that an open subset U of a complex surface can be topologically isotoped so that the complex structure it inherits as an open subset is Stein. The isotopy cannot be made smooth in general, and there are frequently uncountably many choices for the resulting smooth structure on U. This theorem is a manifestation of Freedman's general principle that high-dimensional differential topology tends to work up to homeomorphism in dimension 4, as applied to Eliashberg's work on higher-dimensional Stein manifolds. By adapting Freedman's methods (Casson handles, gropes and reimbedding), one also obtains more detailed results. A tame 2-complex in a complex surface can be topologically ambiently isotoped so that it is smooth except at one point on each 2-cell, and has a system of Stein neighborhoods indexed by a Cantor set. These neighborhoods are homeomorphic to each other but frequently pairwise nondiffeomorphic.