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THE INNER EQUATION FOR GENERIC ANALYTIC UNFOLDINGS OF THE HOPF-ZERO SINGULARITY

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Abstract. A classical problem in the study of the (conservative) unfoldings of the so called hopf-zero bifurcation, is the computation of the splitting of a heteroclinic connection which exists in the symetric normal form along the z-axis. In this paper we derive the inner system associated to this singular problem, which is independent on the unfolding parameter. We prove the existence of two solutions of this system related with the stable and unstable manifolds of the unfolding, and we give an asymptotic formula for their difference. We check that the results in this work agree with the ones obtained in the regular case by the authors.

1. Introduction.

We consider the family of autonomous differential equations in $\mathbb{R}^3$ given by

$$\begin{aligned} \frac{d\phi}{d\tau} &= -\eta\phi - (\alpha + c\eta)\,\mathrm{i}\,\phi + \varepsilon F_1(\phi,\varphi,\eta) \\ \frac{d\varphi}{d\tau} &= -\eta\varphi + (\alpha + c\eta)\,\mathrm{i}\,\varphi + \varepsilon F_2(\phi,\varphi,\eta) \\ \frac{d\eta}{d\tau} &= \eta^2 + b\phi\varphi + \varepsilon H(\phi,\varphi,\eta) \end{aligned} \tag{1.1}$$

where $(F_1, F_2, H)(\xi, \eta, \zeta) = O(\|(\xi, \eta, \zeta)\|^3)$ are analytic functions on the open ball $B(r_0) := \{(x, y, z) \in \mathbb{C}^3 : \|(x, y, z)\| < r_0\}$ for some $r_0 > 0$. The parameter ε is not necessarily small. In fact, the results given in this paper are rigourously proved for any value of ε, in particular for $\varepsilon = 1$.

It is straightforward to see that, when $F_1 = F_2 = H = 0$, this system has a particular solution, $\Psi(\tau) = (\phi(\tau), \varphi(\tau), \eta(\tau))$ with $\phi(\tau) = \varphi(\tau) = 0$ and $\eta(\tau) = -\frac{1}{\tau}$, verifying that $\lim_{\mathrm{Re}\,\tau\to\pm\infty} \Psi(\tau) = 0$.

Our goal is to prove the existence and useful properties of special solutions of the full system (1.1) $\Psi^{\pm}(\tau, \varepsilon) = (\phi^{\pm}(\tau, \varepsilon), \varphi^{\pm}(\tau, \varepsilon), \eta^{\pm}(\tau, \varepsilon))$ which will be defined in some regions of the complex plane and will satisfy the asymptotic condition

$$\lim_{\mathrm{Re}\,\tau\to\pm\infty} \Psi^{\pm}(\tau, \varepsilon) = 0.$$

We will give an asymptotic formula for its difference $\Psi^-(\tau, \varepsilon) - \Psi^+(\tau, \varepsilon)$ as $\mathrm{Im}\,\tau \to -\infty$.

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1.1. **Motivation.** As we will see in this section, the origin of system (1.1) can be found in the study of the analytic unfoldings of the so called Hopf-zero singularity. More concretely, let us consider a vector field in $\mathbb{R}^3$ which has the origin as a critical point and, for some positive α^*, the eigenvalues of the linear part at the origin are $0, \pm\alpha^*\,\mathrm{i}$. If we assume that the linear part of this vector field is in Jordan normal form it will be given by

$$\begin{pmatrix} 0 & \alpha^* & 0 \\ -\alpha^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \alpha^* > 0. \tag{1.2}$$

The unfoldings of this singularity in the conservative case and all the different behaviour these families can present have been broadly studied [Tak73a, Tak74, Tak73b, Guc81, BV84, AMF$^+$03, FGRLA02, DI98, GH83, CK04, LTW05]. The standard way to proceed in the study of these unfoldings, is to use the normal form theory to write the vector field as simple as possible up to some order and then to study the effects of the non symmetric terms in the dynamics. In our case, we consider X_μ a family of conservative vector fields on $\mathbb{R}^3$ such that X_0 has the origin as a critical point with linear part (1.2). After the normal form procedure up to order two, we obtain that the vector field X_μ in the new coordinates $(\bar{x}, \bar{y}, \bar{z})$ takes the form

$$\begin{aligned}
\frac{d\bar{x}}{ds} &= \bar{x}\,(A_2(\mu) + A_4(\mu)\bar{z}) + \bar{y}\,(A_1(\mu) + A_3(\mu)\bar{z}) + O_3(\bar{x}, \bar{y}, \bar{z}, \mu) \\
\frac{d\bar{y}}{ds} &= -\bar{x}\,(A_1(\mu) + A_3(\mu)\bar{z}) + \bar{y}\,(A_2(\mu) + A_4(\mu)\bar{z}) + O_3(\bar{x}, \bar{y}, \bar{z}, \mu) \\
\frac{d\bar{z}}{ds} &= B_1(\mu) - 2A_2(\mu)\bar{z} + B_3(\mu)(\bar{x}^2 + \bar{y}^2) - A_4(\mu)\bar{z}^2 + O_3(\bar{x}, \bar{y}, \bar{z}, \mu)
\end{aligned}$$

where $A_1(0) = \alpha^*$, $A_2(0) = B_1(0) = 0$. And moreover, after some scaling of the parameters we can assume that $\partial_\mu B_1(0) = -1$. To simplify the notation we call $a_j = A_j(0)$, $b_j = B_j(0)$, for $j = 3, 4$.

When $\mu > 0$, and $a_4 < 0$, we perform the scaling $\bar{x} = (\delta/\sqrt{-a_4})x$, $\bar{y} = (\delta/\sqrt{-a_4})y$, $\bar{z} = (\delta/\sqrt{-a_4})z$, $\delta = \sqrt{\mu}$ and the change of time $t = \sqrt{-a_4}\delta s$. Then the system becomes:

$$\begin{aligned}
\frac{dx}{dt} &= -xz + \left(\frac{\alpha}{\delta} + cz\right) y + \delta^p f(\delta x, \delta y, \delta z, \delta) \\
\frac{dy}{dt} &= -yz - \left(\frac{\alpha}{\delta} + cz\right) x + \delta^p g(\delta x, \delta y, \delta z, \delta) \\
\frac{dz}{dt} &= -1 + b(x^2 + y^2) + z^2 + \delta^p h(\delta x, \delta y, \delta z, \delta)
\end{aligned} \tag{1.3}$$

where $\alpha = \frac{\alpha^*}{\sqrt{-a_4}}$, $c = \frac{a_3}{\sqrt{-a_4}}$, $p = -2$, and $f, g, h = O(\|(\delta x, \delta y, \delta z, \delta)\|^3)$ are analytic functions in $B(r_0)$.

In [BS] the authors studied this system in the perturbative case $p > -2$ and they gave a rigorous proof of the breakdown of a heteroclinic orbit (located at $x = y = 0$) that exists if we consider only the terms coming from the normal form, that is, the case $f = g = h = 0$. The proof consisted in validating that the first order perturbation theory, that in this case was explicitly given by a Melnikov function, provided the correct prediction even if the Melnikov function (and the corresponding distance between the invariant manifolds) was exponentially small with respect to the parameter δ.

In this paper we will give some partial results to cover the case $p = -2$. To this end, and in order to compare with the results already obtained when $p > -2$, we add an extra parameter ε, not necessarily small, and we consider the system:

$$\begin{aligned} \frac{dx}{dt} &= -xz + \left(\frac{\alpha}{\delta} + cz\right) y + \varepsilon\delta^{-2} f(\delta x, \delta y, \delta z, \delta) \\ \frac{dy}{dt} &= -yz - \left(\frac{\alpha}{\delta} + cz\right) x + \varepsilon\delta^{-2} g(\delta x, \delta y, \delta z, \delta) \\ \frac{dz}{dt} &= -1 + b(x^2 + y^2) + z^2 + \varepsilon\delta^{-2} h(\delta x, \delta y, \delta z, \delta) \end{aligned} \tag{1.4}$$

Let us observe that, taking $\varepsilon = \delta^{p+2}$ we recover the result given in [BS] for $p > -2$ when ε is small and we deal with the case corresponding to a generic unfolding taking $\varepsilon = 1$.

Let us observe that, if $\varepsilon = 0$, system (1.4) has two fixed points given by $S_\pm = (0, 0, \pm 1)$ with eigenvalues $\mp 1 + |\frac{\alpha}{\delta} \pm c|\,\mathrm{i}$, $\mp 1 - |\frac{\alpha}{\delta} \pm c|\,\mathrm{i}$ and ± 2. Moreover, one branch of the one-dimensional unstable manifold of S_+ and one branch of the one-dimensional stable manifold of S_- coincide giving rise to a heteroclinic orbit between them which can be parameterized by

$$\sigma_0(t) = (0, 0, -\tanh t), \qquad \lim_{t\to\pm\infty} \sigma_0(t) = S_\mp. \tag{1.5}$$

The following result assures that for any $\varepsilon > 0$, system (1.4) has two fixed points of saddle-focus type.

Lemma 1.1. *Given any $\varepsilon_0 > 0$, there exists δ_0 small enough such that if $0 < \delta < \delta_0$ and $|\varepsilon| < \varepsilon_0$, system (1.4) has two fixed points $S_\pm(\delta, \varepsilon)$ of saddle-focus type such that $S_+(\delta, \varepsilon)$ has a one-dimensional unstable manifold and $S_-(\delta, \varepsilon)$ has a stable one. We call them $W^{\mathrm{u,s}}$ respectively.*

Moreover, for any $\nu < 1/3$ there exists $0 < \delta_1 \le \delta_0$ such that there are no other fixed points of (1.4) in the closed ball $B(\delta^{-\nu})$ if $0 < \delta < \delta_1$.

Proof. It is straightforward since we only need to consider the function

$$P(x, y, z, \delta) = \begin{pmatrix} -xz\delta + (\alpha + \delta cz)y + \varepsilon\delta^{-1} f(\delta x, \delta y, \delta z, \delta) \\ -yz\delta - (\alpha + \delta cz)x + \varepsilon\delta^{-1} g(\delta x, \delta y, \delta z, \delta) \\ -1 + b(x^2 + y^2) + z^2 + \varepsilon\delta^{-2} h(\delta x, \delta y, \delta z, \delta) \end{pmatrix}$$

and apply adequately the implicit function theorem as it was done in [BS]. □

Once we know that the critical points $S_\pm(\delta, \varepsilon)$ exist and are hyperbolic, it is a natural question to ask if their one-dimensional unstable and stable manifolds W^{u} and W^{s} are either still coincident or they split.

In [BS], an asymptotic formula for the distance between W^{u} and W^{s} when they encounter the plane $z = 0$ was obtained when $p > -2$. This formula showed that this distance is exponentially small, in fact it is $O(\delta^p\, \mathrm{e}^{-\pi|\alpha|/(2\delta)})$. To prove this fact, it was crucial to obtain good parameterizations of the stable and unstable manifolds, $W^{\mathrm{u,s}}$, in a complex domain which reaches a neighborhood of order δ of the singularities $\pm\,\mathrm{i}\,\pi/2$ of the heteroclinic connection (1.5) of the unperturbed system (system (1.3) with $f = g = h = 0$).

These parameterizations $x^{\mathrm{u,s}}(t), y^{\mathrm{u,s}}(t), z^{\mathrm{u,s}}(t)$ of $W^{\mathrm{u,s}}$, behave as

$$x^{\mathrm{u,s}}(t), y^{\mathrm{u,s}}(t) \sim C\delta^{p+4} \left| t \mp \mathrm{i}\,\frac{\pi}{2} \right|^{-3}, \; z^{\mathrm{u,s}}(t) + \tanh t \sim C\delta^{p+3} \log\delta \left| t \mp \mathrm{i}\,\frac{\pi}{2} \right|^{-2} \tag{1.6}$$

as $t \sim \pm\,\mathrm{i}\,\pi/2$.

Even if the results in [BS] are only valid for $p > -2$, these estimates indicate that if $|t \mp \mathrm{i}\,\pi/2| = O(\delta)$, then for $p=-2$,

$$x^{\mathrm{u,s}}(t), y^{\mathrm{u,s}}(t), z^{\mathrm{u,s}}(t) \sim O(\delta^{-1}).$$

So, in the case $p=-2$, when we evaluate the vector field (1.3) at $x^{\mathrm{u,s}}, y^{\mathrm{u,s}}, z^{\mathrm{u,s}}$, all the terms become of order $O(\delta^{-2})$ and the system is not a perturbation of the case $f=g=h=0$ anymore.

System (1.1) comes from (1.4) (which is (1.3) in the case $\varepsilon = 1$) performing adequate changes of coordinates for studying the behavior of the solution when $|t \mp \mathrm{i}\,\pi/2| = O(\delta)$ and taking into account the dominant terms in δ. Concretely, for studying the behavior of the one dimensional stable and unstable manifolds of system (1.4) in a neighborhood of the singularity $t = \mathrm{i}\,\pi/2$, taking into account (1.6), we perform the change of coordinates $(\phi,\varphi,\eta) = C_\delta(x,y,z)$ given by

$$\phi = \delta(x+\mathrm{i}\,y), \qquad \varphi = \delta(x - \mathrm{i}\,y), \qquad \eta = \delta z, \qquad \tau = \frac{t-\mathrm{i}\,\pi/2}{\delta}$$

and we obtain the system

$$\begin{aligned}
\frac{d\phi}{d\tau} &= \big(-(\alpha + c\eta)\,\mathrm{i} - \eta\big)\phi + \varepsilon \tilde{F}_1(\phi,\varphi,\eta,\delta)\\
\frac{d\varphi}{d\tau} &= \big((\alpha + c\eta)\,\mathrm{i} - \eta\big)\varphi + \varepsilon \tilde{F}_2(\phi,\varphi,\eta,\delta)\\
\frac{d\eta}{d\tau} &= -\delta^2 + b\phi\varphi + \eta^2 + \varepsilon \tilde{H}(\phi,\varphi,\eta,\delta)
\end{aligned} \tag{1.7}$$

where

$$\begin{aligned}
\tilde{F}_1(\phi,\varphi,\eta,\delta) &= f(C_\delta^{-1}(\phi,\varphi,\eta),\delta) + \mathrm{i}\,g(C_\delta^{-1}(\phi,\varphi,\eta),\delta),\\
\tilde{F}_2(\phi,\varphi,\eta,\delta) &= f(C_\delta^{-1}(\phi,\varphi,\eta),\delta) - \mathrm{i}\,g(C_\delta^{-1}(\phi,\varphi,\eta),\delta),\\
\tilde{H}(\phi,\varphi,\eta,\delta) &= h(C_\delta^{-1}(\phi,\varphi,\eta),\delta).
\end{aligned}$$

Then, for studying the behavior of $W^{\mathrm{u,s}}$ when t is close to $\mathrm{i}\,\pi/2$, it is natural to consider, as a first approximation, system (1.7) with $\delta = 0$ which is system (1.1) under consideration in this paper taking $F_i(\phi,\varphi,\eta) = \tilde{F}_i(\phi,\varphi,\eta,0)$, and $H(\phi,\varphi,\eta) = \tilde{H}(\phi,\varphi,\eta,0)$. Using the language of asymptotic methods, one can say that system (1.1) is the "inner" system associated to the invariant manifolds of system (1.4). Moreover, one expects that, around the singularities of the heteroclinic connection, the stable and unstable one dimensional manifolds of system (1.4) will behave as

$$x^{\mathrm{u,s}}(\tau), y^{\mathrm{u,s}}(\tau) \sim \frac{1}{\delta\tau^3}, \qquad z^{\mathrm{u,s}}(\tau) \sim \frac{1}{\delta\tau}.$$

Thus, after the change of variables, they will be well approximated by special solutions, $\Psi^{\pm}$ of system (1.1) having the asymptotic behavior

$$\lim_{\mathrm{Re}\,\tau\to\pm\infty} \Psi^{\pm}(\tau) = 0, \qquad \mathrm{Im}\,\tau < 0.$$

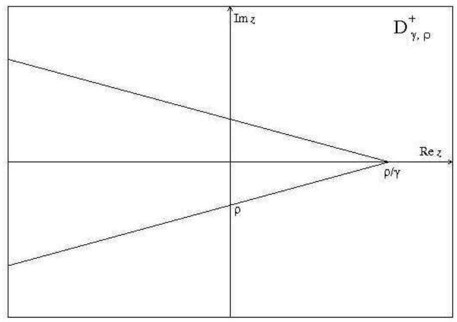
1.2. **Main results.** This Section is devoted to introduce the main results and the notation we will use along this work.

Given $\gamma,\rho,\varepsilon_0 > 0$ and $\nu \in \mathbb{R}$, we define the domains $B(\varepsilon_0) = \{\varepsilon \in \mathbb{C} : |\varepsilon| < \varepsilon_0\}$,

$$D^{+}_{\gamma,\rho} = \{z \in \mathbb{C} : |\,\mathrm{Im}\, z| > -\gamma\,\mathrm{Re}\, z + \rho\}, \qquad D^{-}_{\gamma,\rho} = -D^{+}_{\gamma,\rho},$$

$$E_{\gamma,\rho} = D^{+}_{\gamma,\rho} \cap D^{-}_{\gamma,\rho} \cap \{z \in \mathbb{C} : \mathrm{Im}\, z < 0\},$$

that is



and the functional spaces

$$\mathcal{X}^{\pm}_{\nu,\gamma,\rho} = \{\psi : D^{\pm}_{\gamma,\rho} \times B(\varepsilon_0) \to \mathbb{C},\, \psi \text{ analytic}, \sup_{(z,\varepsilon)\in D^{\pm}_{\gamma,\rho}\times B(\varepsilon_0)} |z^{\nu}\psi(z,\varepsilon)| < +\infty\}$$

$$\mathcal{Y}_{\nu,\gamma,\rho} = \{\psi : E_{\gamma,\rho} \times B(\varepsilon_0) \to \mathbb{C},\, \psi \text{ analytic}, \sup_{(z,\varepsilon)\in E_{\gamma,\rho}\times B(\varepsilon_0)} |z^{\nu}\psi(z,\varepsilon)| < +\infty\}$$

We endow $\mathcal{X}^{\pm}_{\nu,\gamma,\rho}$ and $\mathcal{Y}_{\nu,\gamma,\rho}$ with the norms

$$\|\psi\|^{\pm}_{\nu} = \sup_{(z,\varepsilon)\in D^{\pm}_{\gamma,\rho}\times B(\varepsilon_0)} |z^{\nu}\psi(z,\varepsilon)|, \qquad \|\psi\|_{\nu} = \sup_{(z,\varepsilon)\in E_{\gamma,\rho}\times B(\varepsilon_0)} |z^{\nu}\psi(z,\varepsilon)|$$

respectively and they become Banach spaces.

We also consider the norms

$$\|(\phi,\varphi)\|^{\pm}_{\nu,\times} = \max\{\|\phi\|^{\pm}_{\nu}, \|\varphi\|^{\pm}_{\nu}\}, \qquad \|(\phi,\varphi)\|_{\nu,\times} = \max\{\|\phi\|_{\nu}, \|\varphi\|_{\nu}\}$$

defined on the product spaces $\mathcal{X}^{\pm}_{\nu,\gamma,\rho} \times \mathcal{X}^{\pm}_{\nu,\gamma,\rho}$ and $\mathcal{Y}_{\nu,\gamma,\rho} \times \mathcal{Y}_{\nu,\gamma,\rho}$ respectively.

We also use the notation $\|\cdot\|$ to indicate the supremum norm. As usual we will denote by π^i the projection on the i-component and by $\pi^{i,j}$ the projection on the i,j-components.

Theorem 1.2. *Given $\gamma, \varepsilon_0 > 0$ there exists ρ big enough such that system (1.1) has two solutions $\Psi^{\pm} \in \mathcal{X}^{\pm}_{3,\gamma,\rho} \times \mathcal{X}^{\pm}_{3,\gamma,\rho} \times \mathcal{X}^{\pm}_{1,\gamma,\rho}$ respectively satisfying*

$$\sup_{(\tau,\varepsilon)\in D^{\pm}_{\gamma,\rho}\times B(\varepsilon_0)} |\tau^2(\log\tau)^{-1}(\pi^3\Psi^{\pm}(\tau,\varepsilon) - \tau^{-1})| < \infty.$$

Let $\Delta\Psi = \Psi^- - \Psi^+$. There exist $C(\varepsilon)$, an analytic function on $B(\varepsilon_0)$, and $\xi(\tau,\varepsilon)$, an analytic function on $E_{\gamma,\rho} \times B(\varepsilon_0)$ satisfying $\lim_{\operatorname{Im}\tau\to-\infty} \xi(\tau,\varepsilon) = 0$, such that

$$\begin{pmatrix} \pi^{1,2}\Delta\Psi(\tau,\varepsilon) \\ \tau^2\pi^3\Delta\Psi(\tau,\varepsilon) \end{pmatrix} = \tau\, \mathrm{e}^{-\,\mathrm{i}(|\alpha|\tau - c\log\tau)}\, \varepsilon(C(\varepsilon) + \xi(\tau,\varepsilon)), \qquad \tau \in E_{\gamma,\rho}.$$

We also have that $\pi^{1,2}\xi = O(\tau^{-1}\log\tau)$.

Moreover $C(\varepsilon) \neq 0$ if and only if $\Delta\Psi \neq 0$ and the constant $C(\varepsilon)$ satisfies that $\pi^2 C(\varepsilon) = 0$ if $\alpha > 0$ and, analogously, $\pi^1 C(\varepsilon) = 0$ if $\alpha < 0$.

Let

$$m_1(\tau) = \tau^{-1-\mathrm{i}\,c} F_1(0,0,-\tau^{-1},0) = \sum_{n\geq 3} \frac{m^1_n}{\tau^{n+1+\mathrm{i}\,c}},$$

$$m_2(\tau) = \tau^{-1+\mathrm{i}\,c} F_2(0,0,-\tau^{-1},0) = \sum_{n\geq 3} \frac{m^2_n}{\tau^{n+1-\mathrm{i}\,c}}$$

and their Borel transform

$$\hat{m}_1(\zeta) = \sum_{n\geq 3} m_n^1 \frac{\zeta^{n+\mathrm{i}\,c}}{\Gamma(n+1+\mathrm{i}\,c)}, \qquad \hat{m}_2(\zeta) = \sum_{n\geq 3} m_n^2 \frac{\zeta^{n-\mathrm{i}\,c}}{\Gamma(n+1-\mathrm{i}\,c)}.$$

Then we have that

$$\begin{aligned} \pi^1 C(0) &= 2\pi\,\mathrm{i}\,\hat{m}_1(\mathrm{i}\,\alpha), \qquad \alpha > 0 \\ \pi^2 C(0) &= 2\pi\,\mathrm{i}\,\hat{m}_2(-\mathrm{i}\,\alpha), \qquad \alpha < 0 \end{aligned}$$

Remark 1.3. *In* [BS] *the authors found the following asymptotic formula for the difference of the stable and unstable manifolds of $S_\pm$ in system (1.4), as a function of t, for $\alpha > 0$:*

$$\delta((x^u(t) - x^s(t)) + \mathrm{i}(y^u(t) - y^s(t))) = \\ \cosh t\, \mathrm{e}^{-\mathrm{i}\,\frac{\alpha}{\delta}(t - \mathrm{i}\,\frac{\pi}{2})}\, \mathrm{e}^{\mathrm{i}\,c\ln\frac{\cosh t}{\delta}}\, 2\pi\delta^{p+1}\hat{m}_1(\mathrm{i}\,\alpha)\,\mathrm{e}^{c\frac{\pi}{2}} + O(\delta^{p+2}\,\mathrm{e}^{-\frac{\alpha\pi}{2\delta}})$$

If we take $\varepsilon = \delta^{p+2}$, it is straightforward to see that the main term of this formula when $t = \mathrm{i}\,\frac{\pi}{2} + \delta\tau$ is given by $\pi^1\Delta\Psi(\tau,\varepsilon)$ in Theorem 1.2.

However, a rigorous asymptotic formula for $(x^u(t) - x^s(t)) + \mathrm{i}(y^u(t) - y^s(t))$ is still needed in the case $p = -2$. To this end, the authors will use complex matching to validate that the invariant manifolds $W^{u,s}$ are well approximated by solutions $\Psi^\pm$ when t is close to $\pm\,\mathrm{i}\,\pi/2$ respectively and, consequently their difference can be obtained, up to first order, by suitable combinations of the differences of Ψ^+ and Ψ^-.

1.3. **Preliminaries.** For technical reasons, we perform the change of variables given by $(\phi, \varphi, \eta) = (\phi, \varphi, -s^{-1})$ and we obtain that system (1.1) can be expressed as:

$$\begin{aligned} \frac{d\phi}{d\tau} &= \frac{1}{s}\phi - \left(\alpha - \frac{c}{s}\right)\mathrm{i}\,\phi + \varepsilon F_1(\phi,\varphi,-s^{-1}) \\ \frac{d\varphi}{d\tau} &= \frac{1}{s}\varphi + \left(\alpha - \frac{c}{s}\right)\mathrm{i}\,\varphi + \varepsilon F_2(\phi,\varphi,-s^{-1}) \\ \frac{ds}{d\tau} &= 1 + s^2(b\phi\varphi + \varepsilon H(\phi,\varphi,-s^{-1})). \end{aligned} \tag{1.8}$$

We write $F = (F_1, F_2)$ and

$$A(s) = \begin{pmatrix} -(\alpha - cs^{-1})\,\mathrm{i} + s^{-1} & 0 \\ 0 & (\alpha - cs^{-1})\,\mathrm{i} + s^{-1} \end{pmatrix}. \tag{1.9}$$

Let us consider system

$$\begin{pmatrix} \phi' \\ \varphi' \end{pmatrix} = \frac{1}{1 + s^2(b\phi\varphi + \varepsilon H(\phi,\varphi,-s^{-1}))}\left(A(s)\begin{pmatrix} \phi \\ \varphi \end{pmatrix} + \varepsilon F(\phi,\varphi,-s^{-1})\right) \tag{1.10}$$

with $' = \frac{d}{ds}$, which is is formally equivalent to (1.8). From now on we will work with this system instead of (1.8). In Section 3.2 we will check that both are equivalent.

Theorem 1.4. *Given $\gamma, \varepsilon_0 > 0$, there exists ρ big enough such that system (1.10) has two solutions $(\phi^\pm, \varphi^\pm)$ belonging to $\mathcal{X}^\pm_{3,\gamma,\rho} \times \mathcal{X}^\pm_{3\gamma,\rho}$ respectively. These solutions are the only ones satisfying the asymptotic condition:*

$$\lim_{\operatorname{Re} s \to \pm\infty} (\phi^\pm(s,\varepsilon), \varphi^\pm(s,\varepsilon)) = 0.$$

Let $\Delta\Phi = (\phi^- - \phi^+, \varphi^- - \varphi^+)$. *There exists* $\tilde{\xi} \in \mathcal{Y}_{1,\gamma,\rho} \times \mathcal{Y}_{2,\gamma,\rho}$ *and an analytic function on* $B(\varepsilon_0)$, $C(\varepsilon)$, *such that*

$$\Delta\Phi(s,\varepsilon) = s\,\mathrm{e}^{-\,\mathrm{i}(|\alpha|s-(c+\varepsilon\alpha h_0)\log s)}\,\varepsilon(C(\varepsilon)+\tilde{\xi}(s,\varepsilon)) \tag{1.11}$$

with $h_0 = \lim_{s\to+\infty} s^3 H(0,0,-s^{-1})$. *In addition,* $C(\varepsilon) \neq 0$ *if and only if* $\Delta\Phi \neq 0$ *and* $\pi^2 C(\varepsilon) = 0$ *if* $\alpha > 0$ *and* $\pi^1 C(\varepsilon) = 0$ *if* $\alpha < 0$.

We also have that

$$\begin{aligned} \pi^1 C(0) &= 2\pi\,\mathrm{i}\,\hat{m}_1(\mathrm{i}\,\alpha), \qquad \alpha > 0 \\ \pi^2 C(0) &= 2\pi\,\mathrm{i}\,\hat{m}_2(-\,\mathrm{i}\,\alpha), \qquad \alpha < 0. \end{aligned}$$

The main part of the paper is devoted to prove Theorem 1.4. The proof of this result is decomposed in two main parts. The first one deals with the existence and appropriate properties of solutions $\phi^\pm, \varphi^\pm$ of system (1.10) and it is done in Section 2. The second step is related to the asymptotic expression given in (1.11) and it is postposed to Section 3. We will recover Theorem 1.2 from Theorem 1.4 in Section 3.2.

2. **Existence of solutions** $(\phi^\pm, \varphi^\pm)$.

In this section we will prove the existence and useful properties of solutions $(\phi^\pm, \varphi^\pm)$ of system (1.10) having the asymptotic property $\lim_{\mathrm{Re}\, s\to\pm\infty}(\phi^\pm, \varphi^\pm) = 0$.

In Subsection 2.1 we introduce some notation and the set up we will work in. More concretely we will reduce our problem to a fixed point problem in adequate Banach spaces. At the end of this Subsection, we enunciate the rigorous statement about solutions $(\phi^\pm, \varphi^\pm)$ which we will prove in Subsections 2.2 and 2.3.

2.1. **Set up.**

Now we are going to write system (1.10) in a more appropriate way.

Let $h_0 = \lim_{s\to\infty} H(0,0,-s^{-1})s^{-3}$ as in Theorem 1.4. We decompose the function $s^2(b\phi\varphi + \varepsilon H)$ in the form

$$s^2(b\phi\varphi + \varepsilon H(\phi,\varphi,-s^{-1})) = \varepsilon h_0 s^{-1} + \bar{H}(\phi,\varphi,-s^{-1},\varepsilon) \tag{2.1}$$

with $\bar{H}(0,0,-s^{-1},\varepsilon) = \varepsilon s^2\big(H(0,0,-s^{-1}) - h_0 s^{-3}\big) = O(|s|^{-2})$. We also introduce

$$\begin{aligned} \mathcal{R}(\phi,\varphi)(s,\varepsilon) &= \varepsilon F(\phi,\varphi,-s^{-1})(1+\varepsilon h_0 s^{-1} + \bar{H}(\phi,\varphi,-s^{-1},\varepsilon))^{-1} \\ &\quad + A(s)\big[(1+\varepsilon h_0 s^{-1} + \bar{H}(\phi,\varphi,-s^{-1},\varepsilon))^{-1} - (1+\varepsilon h_0 s^{-1})^{-1}\big]\begin{pmatrix}\phi\\ \varphi\end{pmatrix}. \end{aligned} \tag{2.2}$$

Hence, using decomposition (2.1), it is clear that system (1.10) can be written in the form

$$\frac{d}{ds}\begin{pmatrix}\phi\\ \varphi\end{pmatrix} = (1+\varepsilon h_0 s^{-1})^{-1}A(s)\begin{pmatrix}\phi\\ \varphi\end{pmatrix} + \mathcal{R}(\phi,\varphi)(s,\varepsilon). \tag{2.3}$$

Lemma 2.1. *The linear system*

$$\frac{d}{ds}\begin{pmatrix}\phi\\ \varphi\end{pmatrix} = (1+\varepsilon h_0 s^{-1})^{-1}A(s)\begin{pmatrix}\phi\\ \varphi\end{pmatrix} \tag{2.4}$$

has a fundamental matrix of the form

$$M(s) = s(1+\varepsilon h_0 s^{-1})\,\mathrm{diag}\left(\mathrm{e}^{-\,\mathrm{i}(\alpha s+\beta(s,\varepsilon))}, \mathrm{e}^{\mathrm{i}(\alpha s+\beta(s,\varepsilon))}\right) \tag{2.5}$$

where $\beta(s,\varepsilon) = -(c+\varepsilon\alpha h_0)\log\big(s(1+\varepsilon h_0 s^{-1})\big)$.

To prove Lemma 2.1 is straightforward.

Remark 2.2. *We recall that we are looking for solutions $\phi^\pm, \varphi^\pm$ defined on $D^\pm_{\gamma,\rho}$ respectively. Hence, when we deal with the $-$ case, in the definition of β we choose a determination of logarithm defined in $\mathbb{C}\backslash\{u\in\mathbb{C} : \operatorname{Im} u = 0,\ \operatorname{Re} u \geq 0\}$ and, analogously, we take the logarithm in the $+$ case by a determination defined in $\mathbb{C}\backslash\{u\in\mathbb{C} : \operatorname{Im} u = 0,\ \operatorname{Re} u \leq 0\}$.*

Next we write the functional equation that $(\phi^\pm, \varphi^\pm)$ have to satisfy. We introduce $\Phi^\pm = (\phi^\pm, \varphi^\pm)$ and we note that, since $\Phi^\pm$ are solutions of system (2.3), by Lemma 2.1 they can be expressed as

$$\Phi^\pm(s) = M(s)\left(K^\pm + \int_{s_0^\pm}^{s} M(t)^{-1}\mathcal{R}(\Phi^\pm)(t,\varepsilon)\,dt\right)$$

with $K^\pm \in \mathbb{C}^2$ and $s_0^\pm \in D^\pm_{\gamma,\rho}$ respectively. Now we impose that

$$\lim_{\operatorname{Re} s\to\pm\infty} \Phi^\pm(s,\varepsilon) = 0,$$

hence, since $\|M(s)\| \to +\infty$ as $\operatorname{Re} s \to \pm\infty$, the constants $K^\pm$ are determined by $K^\pm = -\int_{s_0^\pm}^{\pm\infty} M(t)^{-1}\mathcal{R}(\Phi^\pm)(t,\varepsilon)\,dt$ and then $\Phi^\pm$ satisfy the fixed point equation

$$\Phi^\pm(s) = M(s)\int_{\pm\infty}^{s} M(t)^{-1}\mathcal{R}(\Phi^\pm)(t,\varepsilon)\,dt. \tag{2.6}$$

Finally we observe that, since we are looking for bounded solutions of system (2.3) and $\|M(s)^{-1}\| \to 0$ as $|\operatorname{Re} s| \to \infty$, by Cauchy's theorem, the fixed point equation (2.6) is equivalent to

$$\Phi^\pm(s) = M(s)\int_{\pm\infty}^{0} M(s+t)^{-1}\mathcal{R}(\Phi^\pm)(s+t,\varepsilon)\,dt. \tag{2.7}$$

We introduce the linear operators

$$\mathcal{B}^\pm(\psi)(s) = M(s)\int_{\pm\infty}^{0} M(s+t)^{-1}\psi(s+t)\,dt$$

and we stress that the fixed point equation (2.7) can be written as

$$\Phi^\pm = \mathcal{F}^\pm(\Phi^\pm) := \mathcal{B}^\pm\circ\mathcal{R}(\Phi^\pm), \tag{2.8}$$

where $\mathcal{R}$ was defined in (2.2).

The remaining part of this section is devoted to prove the following proposition:

Proposition 2.3. *Given $\gamma, \varepsilon_0 > 0$, there exists ρ big enough such that system (2.3) has two solutions $\Phi^\pm$ belonging to $\mathcal{X}^\pm_{3,\gamma,\rho}\times\mathcal{X}^\pm_{3,\gamma,\rho}$ of the form $\Phi^\pm = \Phi^\pm_0 + \Phi^\pm_1$ with $\Phi^\pm_0 = \mathcal{B}^\pm\circ\mathcal{R}(0) \in \mathcal{X}^\pm_{3,\gamma,\rho}\times\mathcal{X}^\pm_{3,\gamma,\rho}$ and $\Phi^\pm_1 \in \mathcal{X}^\pm_{4,\gamma,\rho}\times\mathcal{X}^\pm_{4,\gamma,\rho}$. We also have that $\|\Phi^\pm_1\|_{3,\times} < \|\Phi^\pm_0\|_{3,\times}$.*

Moreover they are the unique solutions of (2.3) satisfying the asymptotic condition $\lim_{\operatorname{Re} s\to\pm\infty}\Phi^\pm(s) = 0$.

2.2. **The linear operators $\mathcal{B}^\pm$.** Now we are going to study the linear operators $\mathcal{B}^\pm$. Along this section, if there is no danger of confusion, we will omit the dependence on γ,ρ of the Banach spaces $\mathcal{X}^\pm_{\nu,\gamma,\rho}$, thus we will write them simply as $\mathcal{X}^\pm_\nu$.

Lemma 2.4. *Let $\nu, \gamma, \varepsilon_0 > 0$. There exists ρ big enough such that the operators $\mathcal{B}^\pm : \mathcal{X}_\nu^\pm \times \mathcal{X}_\nu^\pm \to \mathcal{X}_{\nu-1}^\pm \times \mathcal{X}_{\nu-1}^\pm$ are well defined and there exists a constant $C_{\mathcal{B}^\pm}$ depending on $\nu, \gamma, \varepsilon_0, \alpha, c$ such that*

$$\|\mathcal{B}^\pm(\psi)\|_{\nu-1,\times}^\pm \leq C_{\mathcal{B}^\pm} \|\psi\|_{\nu,\times}^\pm, \qquad \text{for all } \psi \in \mathcal{X}_\nu^\pm \times \mathcal{X}_\nu^\pm.$$

In addition,

$$\mathcal{B}^\pm(\psi) \in \mathcal{X}_\nu^\pm \times \mathcal{X}_\nu^\pm, \qquad \text{for all } \psi \in \mathcal{X}_\nu^\pm \times \mathcal{X}_\nu^\pm,\ D\psi \in \mathcal{X}_{\nu+1}^\pm \times \mathcal{X}_{\nu+1}^\pm. \tag{2.9}$$

Here D denotes the derivative with respect to s.

Proof. We prove Lemma 2.4 only in the $-$ case, being the $+$ case analogous. As we pointed out in Remark 2.2 we have chosen a determination of the logarithm defined in $\mathbb{C}\backslash\{u \in \mathbb{C} : \operatorname{Im} z = 0,\ \operatorname{Re} z \geq 0\}$.

We claim that there exists a constant C depending on c, ε_0, α such that, if ρ is big enough, for all $s \in D_{\gamma,\rho}^-$ and $t \leq 0$,

$$\|M(s)M(s+t)^{-1}\| \leq C\frac{|s|}{|s+t|}. \tag{2.10}$$

($\|\cdot\|$ denotes the matricial supremum norm). Indeed, for all $t \leq 0$, $s+t \in D_{\gamma,\rho}^-$ and $\arg(s+t) \in (\arctan(\gamma), 2\pi - \arctan(\gamma))$. Moreover, if $\rho > \varepsilon_0|h_0|\sqrt{1+\gamma^2}$, then $s+t+\varepsilon h_0 \in D^-_{\gamma,\rho-\varepsilon_0|h_0|\sqrt{1+\gamma^2}}$ and we also have that $\arg(s+t+\varepsilon h_0) \in (\arctan(\gamma), 2\pi - \arctan(\gamma))$. Therefore

$$\begin{aligned}\left|\operatorname{Im}\left(\beta(s,\varepsilon) - \beta(s+t,\varepsilon)\right)\right| &= \left|\operatorname{Im}\left((c+\varepsilon\alpha h_0)\log\left(\frac{s+\varepsilon h_0}{s+t+\varepsilon h_0}\right)\right)\right| \\ &= |c+\varepsilon\alpha h_0|\left|\arg(s+\varepsilon h_0) - \arg(s+t+\varepsilon h_0)\right| \\ &\leq 2|2\pi - \arctan(\gamma)| \cdot (|c| + \varepsilon_0|\alpha h_0|).\end{aligned} \tag{2.11}$$

Taking into account definition (2.5) of M, this bound implies that

$$\|M(s)M(t+s)^{-1}\| \leq \frac{|s||1+\varepsilon h_0 s^{-1}|}{|s+t||1+\varepsilon h_0(s+t)^{-1}|}\exp(2|2\pi - \arctan(\gamma)|\cdot(|c|+\varepsilon_0|\alpha h_0|))$$

and (2.10) follows from the above bound taking into account that, if $s \in D_{\gamma,\rho}^-$ and $t \leq 0$, $|s+t| > \rho(1+\gamma^2)^{-1/2}$ and then $|1+\varepsilon h_0 s^{-1}||1+\varepsilon h_0(s+t)^{-1}|^{-1}$ is bounded if ρ is big enough.

Let $\psi \in \mathcal{X}_\nu^- \times \mathcal{X}_\nu^-$. By (2.10), it is clear that, for all $s \in D_{\gamma,\rho}^-$ and $t \leq 0$,

$$\|M(s)M(s+t)^{-1}\psi(s+t)\| \leq C\|\psi\|_{\nu,\times}^- \frac{|s|}{|s+t|^{\nu+1}} \tag{2.12}$$

(here $\|\cdot\|$ denotes the supremum norm in $\mathbb{R}^2$).

We claim that

$$\int_{-\infty}^0 \frac{1}{|s+t|^{\nu+1}}\,dt \leq K_{\nu,\gamma}\frac{1}{|s|^\nu} \qquad \text{if } s \in D_{\gamma,\rho}^-, \tag{2.13}$$

where $K_{\nu,\gamma} = 2(1+\gamma^2)^{\nu/2}\gamma^{-\nu}\int_0^{+\infty}(1+t^2)^{(\nu+1)/2}\,dt$. We can check bound (2.13) by using that, if $\operatorname{Re} s > 0$ and $s \in D_{\gamma,\rho}^-$, $\gamma|s| \leq (1+\gamma^2)^{1/2}|\operatorname{Im} s|$. The case $\operatorname{Re} s \leq 0$ is obvious. Hence, using (2.12) and (2.13) to bound $\mathcal{B}^-(\psi)$ we have that, for all $s \in D_{\gamma,\rho}^-$,

$$\|\mathcal{B}^-(\psi)(s)\| \leq C\|\psi\|_{\nu,\times}^-|s|\int_{-\infty}^0 \frac{1}{|s+t|^{\nu+1}}\,dt \leq CK_{\nu,\gamma}\|\psi\|_{\nu,\times}^-\frac{1}{|s|^{\nu-1}}$$

and the first part of Lemma 2.4 is proved.

Now we deal with (2.9). Let $\psi \in \mathcal{X}_\nu^- \times \mathcal{X}_\nu^-$ satisfying that $D\psi \in \mathcal{X}_{\nu+1}^- \times \mathcal{X}_{\nu+1}^-$. To simplify the notation we introduce $\overline{A}(s) = A(s)(1+\varepsilon h_0 s^{-1})^{-1}$. Integrating by parts and using that $M^{-1}(s) = -\frac{d}{ds}M^{-1}(s)\overline{A}^{-1}(s)$, we obtain

$$\begin{aligned}
\mathcal{B}^-(\psi) &= -M(s)\int_{-\infty}^0 D_t M^{-1}(s+t)\overline{A}^{-1}(s+t)\psi(s+t)\,dt \\
&= -\overline{A}^{-1}(s)\psi(s) + M(s)\int_{-\infty}^0 M^{-1}(s+t)D_t(\overline{A}^{-1}(s+t)\psi(s+t))\,dt \\
&= -\overline{A}^{-1}(s)\psi(s) + \mathcal{B}^-(D(\overline{A}^{-1}\psi))(s). \qquad (2.14)
\end{aligned}$$

As usual, D denotes the derivative with respect to s. It is clear, by definition (1.9) of A, that all the components of $\overline{A}^{-1}$ belong to $\mathcal{X}_0^-$ and hence $\overline{A}^{-1}\psi \in \mathcal{X}_\nu^- \times \mathcal{X}_\nu^-$. Moreover, a simple computation, shows that all the components of $D\overline{A}^{-1}$ belong to $\mathcal{X}_2^- \times \mathcal{X}_2^-$, hence $D(\overline{A}^{-1}\psi) = D\overline{A}^{-1}\psi + \overline{A}^{-1}D\psi \in \mathcal{X}_{\nu+1}^- \times \mathcal{X}_{\nu+1}^-$ provided that $D\psi \in \mathcal{X}_{\nu+1}^- \times \mathcal{X}_{\nu+1}^-$. The proof is finished, by using equality (2.14) and the fact that $\mathcal{B}^- : \mathcal{X}_{\nu+1}^- \times \mathcal{X}_{\nu+1}^- \to \mathcal{X}_\nu^- \times \mathcal{X}_\nu^-$. □

2.3. **The fixed point equation $\Phi = \mathcal{F}^\pm(\Phi)$.** Let $U \subset \mathbb{C}^n$ be an open neighborhood of 0. Given an analytic function $f : U \subset \mathbb{C}^n \to \mathbb{C}^m$, we introduce the standard notation $f = O_l$ to indicate that f is a function of order l, that is $f(X) = O(\|X\|^l)$.

Lemma 2.5. *Given $r > 0$, let $f : B(r) \subset \mathbb{C}^3 \to \mathbb{C}^m$ be an analytic function such that $f = O_3$. Then, for all $(x, y, z) \in B(r/2)$,*

$$f(x,y,z) - f(0,0,z) = \Delta f(x,y,z)\begin{pmatrix} x \\ y \end{pmatrix}$$

where Δf is a $m \times 2$ matrix with all the components satisfying $\Delta f_{i,j} = O_2$.

Consequently, there exists a constant C, depending on r and f, such that

$$|f(x,y,z) - f(0,0,z)| \le C\|(x,y,z)\|^2\|(x,y)\|.$$

(Here $\|\cdot\|$ denotes the supremum norm).

We enunciate a technical lemma which was proved in [Bal] in a more general setting.

Lemma 2.6. *We fix $\nu \ge 0$, $\gamma, \rho > 0$ and $h \in \mathcal{X}_{\nu,\gamma,\rho}^\pm$. Then for any $l \in \mathbb{N}\backslash\{0\}$,*

$$\partial_s^l h \in \mathcal{X}_{\nu+l,2\gamma,4\rho}^\pm, \qquad \|\partial_s^l h\|_{\nu+l} \le C_l\|h\|_\nu$$

where C_l is a constant depending on l, γ and ρ.

To check this property we use Cauchy's theorem and the fact that there exists a constant $C_{\gamma,\rho}$, depending only on γ and ρ, such that the open ball of center s and radius $C_{\gamma,\rho}|s|$ belongs to $D_{\gamma/2,\rho/4}$ for all $s \in D_{\gamma,\rho}$.

Given $\gamma, \rho > 0$, let $B^\pm(R)$ be the closed ball of radius $R > 0$ and center the origin of $\mathcal{X}_{3,\gamma,\rho}^\pm \times \mathcal{X}_{3,\gamma,\rho}^\pm$. Our purpose is to prove the following lemma

Lemma 2.7. *For any $\gamma > 0$, there exists ρ big enough and $R > 0$ such that the operators $\mathcal{F}^\pm : B^\pm(R) \to B^\pm(R/4)$ are well defined.*

Proof. We prove this lemma only in the $-$ case. The $+$ case can be done in a similar way. Along this proof we omit the $-$ sign in our notation. If there is no danger of confusion we also omit the dependence on ρ, γ and ε in such a way that in the sequel, we will write $\mathcal{X}_\nu$ instead of $\mathcal{X}^-_{\nu,\gamma,\rho}$.

Since $\mathcal{F}$ can be expressed in the form (2.8), we need to study the operator $\mathcal{R}$ defined in (2.2). As F is analytic in $B(r_0)$, we fix ρ_0 such that Lemma 2.4 holds and $\rho_0(1+(\gamma/2)^2)^{-1/2} > 8r_0^{-1}$. We notice that with this choice of ρ_0, if $s \in D_{\gamma/2,\rho_0/4}$, then $|s^{-1}| < r_0/2$.

We fix $\gamma > 0$ and $\rho > 0$ big enough, which we will determinate later on. Let $\Phi = (\phi, \varphi) \in \mathcal{X}_3 \times \mathcal{X}_3$. In order to clarify the notation we introduce

$$\begin{aligned} \bar{h}(\Phi)(s) &= \varepsilon h_0 s^{-1} + \bar{H}(\Phi(s), -s^{-1}), \\ G(\Phi)(s) &= F(\Phi(s), -s^{-1})(1+\bar{h}(\Phi)(s))^{-1}, \end{aligned} \tag{2.15}$$

with $\bar{H}$ defined by (2.1), and we note that

$$\mathcal{R}(\Phi)(s) = \varepsilon G(\Phi)(s) + A(s)\Phi(s)\big[(1+\bar{h}(\Phi)(s))^{-1} - (1+\varepsilon h_0 s^{-1})^{-1}\big]. \tag{2.16}$$

We claim that $\mathcal{R}(0) \in \mathcal{X}_{3,\gamma/2,\rho_0/4} \times \mathcal{X}_{3,\gamma/2,\rho_0/4}$. Indeed, from definition (2.15) of G we have that

$$\begin{aligned} \mathcal{R}(0)(s) = \varepsilon G(0)(s) &= \varepsilon F(0,-s^{-1})(1+\varepsilon h_0 s^{-1} + \bar{H}(0,-s^{-1}))^{-1} \\ &= \varepsilon F(0,-s^{-1})(1+\varepsilon s^2 H(0,-s^{-1}))^{-1} \end{aligned}$$

and since $F(0,-s^{-1}) \in \mathcal{X}_{3,\gamma/2,\rho_0/4} \times \mathcal{X}_{3,\gamma/2,\rho_0/4}$ and $(1+\varepsilon s^2 H(0,-s^{-1}))^{-1} \in \mathcal{X}_{0,\gamma/2,\rho_0/4}$, we have that $\mathcal{R}(0) \in \mathcal{X}_{3,\gamma/2,\rho_0/4} \times \mathcal{X}_{3,\gamma/2,\rho_0/4}$. In addition, by Lemma 2.6, one can deduce that $D\mathcal{R}(0) \in \mathcal{X}_{4,\gamma,\rho_0} \times \mathcal{X}_{4,\gamma,\rho_0}$.

We define the radius

$$R = 8\|\mathcal{B}(\mathcal{R}(0))\|_{3,\times} \tag{2.17}$$

and we observe that, by (2.9) of Lemma 2.4, R is well defined (here the norm $\|\cdot\|_{3,\times}$ is on $\mathcal{X}_{3,\gamma,\rho_0} \times \mathcal{X}_{3,\gamma,\rho_0}$).

Let $\rho \geq \max\{\rho_0, 2^{1/3}R^{1/3}(1+\gamma^2)^{1/2}/r_0^{1/3}\}$ and let $\Phi \in B(R)$. We note that, if $s \in D_{\gamma,\rho}$, then $(\Phi(s), -s^{-1}) \in B(r_0/2) \subset \mathbb{C}^3$, that is $(\Phi(s), -s^{-1})$ belongs to the domain of analyticity of F and $\bar{H}$. Indeed, clearly

$$\|\Phi(s)\| \leq R|s|^{-3} \leq R(1+\gamma^2)^{3/2}\rho^{-3} < \frac{r_0}{2}$$

(as usual, $\|\cdot\|$ denotes the supremum norm). Our goal is to estimate $\|\mathcal{R}(\Phi)(s) - \mathcal{R}(0)(s)\|$ for $s \in D_{\gamma,\rho}$.

Since $H(x,y,z) = O_3$ and taking into account decomposition (2.1), we have that, by Lemma 2.5, for all $s \in D_{\gamma,\rho}$

$$\begin{aligned} |\bar{H}(\Phi(s),-s^{-1}) - \bar{H}(0,-s^{-1})| \leq& |b||s^2\phi(s)\varphi(s)| \\ &+ |\varepsilon s^2(H(\Phi(s),-s^{-1}) - H(0,-s^{-1}))| \\ \leq& |b||s|^{-4}R^2 + C_H(R|s|^{-3} + |s|^{-1})^2 R|s|^{-1} \\ \leq& 2C_H|s|^{-3}R \end{aligned} \tag{2.18}$$

if ρ is big enough. Moreover, since $\bar{H}(0,-s^{-1}) \in \mathcal{X}_2$, there exists K_H such that $|\bar{H}(0,-s^{-1})| \leq K_H|s|^{-2}$ and therefore

$$|\bar{H}(\Phi(s),-s^{-1})| \leq K_H|s|^{-2} + 2C_H|s|^{-3}R. \tag{2.19}$$

Using the fact that $|(1+\xi)^{-1}-(1+\zeta)^{-1}| \le 4|\xi-\zeta|$ if $|\xi|,|\zeta| \le 1/2$ and bounds (2.18) and (2.19), we deduce from definition (2.15) of $\bar h$ that

$$|(1+\bar h(\Phi)(s))^{-1}-(1+\varepsilon h_0 s^{-1})^{-1}| \le 4|s|^{-2}(K_H+2C_H R|s|^{-1})$$
$$|(1+\bar h(\Phi)(s))^{-1}-(1+\bar h(0)(s))^{-1}| \le 8C_H|s|^{-3}R, \tag{2.20}$$

taking ρ big enough to satisfy $|\bar h(\Phi)(s)|<1/2$ and $|\varepsilon h_0 s^{-1}|<1/2$, for all $s\in D_{\gamma,\rho}$.

To bound $\|G(\Phi)(s)-G(0)(s)\|$ we take into account the previous bounds and we use again Lemma 2.5, in order to check that

$$\begin{aligned}\|G(\Phi)(s)-G(0)(s)\| \le &\|F(0,-s^{-1})\||(1+\bar h(\Phi)(s))^{-1}-(1+\bar h(0)(s))^{-1}|\\ &+\|F(\Phi(s),-s^{-1})-F(0,-s^{-1})\||1+\bar h(\Phi)(s)|^{-1}\\ \le &K_G|s|^{-5}R,\end{aligned} \tag{2.21}$$

for some constant K_G, if ρ is big enough. In the last bound we have used that $F(0,-s^{-1})\in\mathcal{X}_3$.

Finally, by definition (2.16) of $\mathcal{R}$, using bounds (2.21) and (2.20) and the fact that $\|A(s)\|\le 2|\alpha|$ if ρ is big enough, we get that for all $s\in D_{\gamma,\rho}$

$$\begin{aligned}\|\mathcal{R}(\Phi)(s)-\mathcal{R}(0)(s)\| &\le |s|^{-5}\big[\varepsilon_0 K_G R+8|\alpha|R(K_H+2C_H R|s|^{-1})\big]\\ &\le |s|^{-4}\frac{R}{8C_{\mathcal{B}}}\end{aligned} \tag{2.22}$$

with $C_{\mathcal{B}}$ the constant given in Lemma 2.4. In the last equality we have taken ρ big enough and we have used that if $s\in D_{\gamma,\rho}$ then $|s|>\rho(1+\gamma^2)^{-1/2}$.

Bound (2.22) implies that $\mathcal{R}(\Phi)-\mathcal{R}(0)\in\mathcal{X}_4\times\mathcal{X}_4$ if $\Phi\in B(R)$ and that $\|\mathcal{R}(\Phi)-\mathcal{R}(0)\|_{4,\times}\le R/(8C_{\mathcal{B}})$. Therefore by Lemma 2.4, taking ρ big enough if necessary, $\mathcal{B}(\mathcal{R}(\Phi))-\mathcal{B}(\mathcal{R}(0))\in\mathcal{X}_3\times\mathcal{X}_3$ and since $\mathcal{B}(\mathcal{R}(0))\in\mathcal{X}_3\times\mathcal{X}_3$, we have that $\mathcal{B}(\mathcal{R}(\Phi))$ also belongs to $\mathcal{X}_3\times\mathcal{X}_3$. Moreover, by definition (2.17) of R and again by bound (2.22), we have that

$$\|\mathcal{B}(\mathcal{R}(\Phi))\|_{3,\times}\le\|\mathcal{B}(\mathcal{R}(0))\|_{3,\times}+\|\mathcal{B}(\mathcal{R}(\Phi)-\mathcal{R}(0))\|_{3,\times}\le\frac{R}{8}+\frac{R}{8}=\frac{R}{4}$$

and then the operator $\mathcal{F}^\pm=\mathcal{B}^\pm\circ\mathcal{R}:B^\pm(R)\to B^\pm(R/4)$ is well defined. □

End of the proof of Proposition 2.3. The existence statement of Proposition 2.3 follows from Lemma 2.7 and

Lemma 2.8 ([Ang93]). *Let E be a complex Banach space, and let $f:B_r\to B_{\theta r}$ be a holomorphic mapping, where $B_\rho=\{x\in E:\|x\|<\rho\}$.*

If $\theta<1/2$, the map $f_{|B_{\theta r}}$ is a contraction, and hence has a unique fixed point in B_r.

Indeed, since the operators $\mathcal{F}^\pm$ are analytic in Φ and $\mathcal{F}^\pm(B^\pm(R))\subset B^\pm(R/4)$, by Lemma 2.8, every $\mathcal{F}^\pm$ is a contraction. Thus $\mathcal{F}^\pm$ have a unique fixed point $\Phi^\pm$ belonging to $B^\pm(R)$, that is, there exist $\Phi^\pm\in B^\pm(R)$ such that $\mathcal{F}^\pm(\Phi^\pm)=\Phi^\pm$.

Moreover we have obtained that $\Phi^\pm=\mathcal{B}^\pm\circ\mathcal{R}(0)+\mathcal{B}^\pm\circ(\mathcal{R}(\Phi^\pm)-\mathcal{R}(0)):=\Phi_0^\pm+\Phi_1^\pm$ with $\Phi_0^\pm\in\mathcal{X}^\pm_{3,\gamma,\rho}\times\mathcal{X}^\pm_{3,\gamma,\rho}$ and $\Phi_1^\pm\in\mathcal{X}^\pm_{4,\gamma,\rho}\times\mathcal{X}^\pm_{4,\gamma,\rho}$. Indeed, we only need to emphasize that, in fact, by (2.22) $\mathcal{R}(\Phi^\pm)-\mathcal{R}(0)\in\mathcal{X}^\pm_{5,\gamma,\rho}\times\mathcal{X}^\pm_{5,\gamma,\rho}$ and apply Lemma 2.4.

Now we deal with the uniqueness. We deal only in the $-$ case. First we will prove that, if Ψ is a solution of system (2.3) defined on $D^-_{\gamma,\rho}$, satisfying the asymptotic condition

$$\lim_{\mathrm{Re}\, s\to -\infty} \Psi(s) = 0, \tag{2.23}$$

then $\Psi \in \mathcal{X}^-_{3,\gamma_0,\rho_0}$ for some $\gamma_0 \geq \gamma$ and $\rho_0 \geq \rho$ big enough. Indeed, by the mean's value theorem

$$\mathcal{R}(\Psi)(s) - \mathcal{R}(0)(s) = \int_0^1 D\mathcal{R}(\lambda\Psi)(s)\, d\lambda \cdot \Psi(s).$$

Henceforth, using that Ψ satisfies the asymptotic condition (2.23), we obtain that, for any $\lambda \in [0,1]$, the function $D\mathcal{R}(\lambda\Psi)(s) \to 0$ as $\mathrm{Re}\, s$ goes to $-\infty$ and thus, taking ρ_1 big enough, $\sup_{s\in D^-_{\gamma,\rho_1}} \|D\mathcal{R}(\lambda\Psi)(s)\| < |\alpha|/4$. This bound implies that, for all $s \in D^-_{\gamma,\rho_1}$,

$$\|\mathcal{R}(\Psi)(s) - \mathcal{R}(0)(s)\| \leq \frac{|\alpha|}{4}\|\Psi(s)\|. \tag{2.24}$$

On the other hand, by Lemma 2.6, we conclude that $\partial_s\Psi \in \mathcal{X}^-_{1,2\gamma,4\rho} \times \mathcal{X}^-_{1,2\gamma,4\rho}$ provided that, by the asymptotic condition (2.23), $\Psi \in \mathcal{X}^-_{0,\gamma,\rho} \times \mathcal{X}^-_{0,\gamma,\rho}$.

Since Ψ is a solution of system (2.3), we have that

$$A(s)(1+\varepsilon h_0 s^{-1})^{-1}\Psi(s) + \mathcal{R}(\Psi)(s) - \mathcal{R}(0)(s) = \frac{d}{ds}\Psi(s) - \mathcal{R}(0)(s).$$

On the one hand we note that, by definition (1.9) of A, $\|A(s)(1+\varepsilon h_0 s^{-1})v\| \geq \frac{|\alpha|}{2}\|v\|$, if $v \in \mathbb{C}^2$. Thus, using bound (2.24), we obtain that for any $s \in D^-_{\gamma,\rho_1}$,

$$\begin{aligned}\|A(s)&(1+\varepsilon h_0 s^{-1})^{-1}\Psi(s) + \mathcal{R}(\Psi)(s) - \mathcal{R}(0)(s)\| \\ &\geq \|A(s)(1+\varepsilon h_0 s^{-1})\Psi(s)\| - \|\mathcal{R}(\Psi)(s) - \mathcal{R}(0)(s)\| \\ &\geq \frac{|\alpha|}{2}\|\Psi(s)\| - \frac{|\alpha|}{4}\|\Psi(s)\| = \frac{|\alpha|}{4}\|\Psi(s)\|.\end{aligned}$$

On the other hand, by Lemma 2.6, for any $s \in D^-_{2\gamma,4\rho}$,

$$\|\frac{d}{ds}\Psi(s) - \mathcal{R}(0)(s)\| \leq \|\partial_s\Psi(s)\| + \|\mathcal{R}(0)(s)\| \leq C_1\|\Psi\|^-_{0,\times}|s|^{-1} + \|\mathcal{R}(0)\|^-_{3,\times}|s|^{-3}.$$

Taking into account the above bounds one concludes that

$$\frac{|\alpha|}{4}\|\Psi(s)\| \leq C_1\|\Psi\|_{0,\times}|s|^{-1} + \|\mathcal{R}(0)\|^-_{3,\times}|s|^{-3}$$

for any $s \in D^-_{\gamma,\rho_1} \cap D^-_{2\gamma,4\rho}$. Consequently, $\Psi \in \mathcal{X}^-_{1,2\gamma,\rho_1} \times \mathcal{X}^-_{1,2\gamma,\rho_1}$, renaming if necessary ρ_1.

Iterating this process, we obtain that, since $\Psi \in \mathcal{X}^-_{1,2\gamma,\rho_1} \times \mathcal{X}^-_{1,2\gamma,\rho_1}$, by Lemma 2.6, for any $s \in D^-_{4\gamma,4\rho_1}$,

$$\|\frac{d}{ds}\Psi(s) - \mathcal{R}(0)(s)\| \leq \|\partial_s\Psi(s)\| + \|\mathcal{R}(0)(s)\| \leq C_2\|\Psi\|^-_{1,\times}|s|^{-2} + \|\mathcal{R}(0)\|^-_{3,\times}|s|^{-3}.$$

and therefore we can conclude that

$$\frac{|\alpha|}{4}\|\Psi(s)\| \leq C_2\|\Psi\|^-_{1,\times}|s|^{-2} + \|\mathcal{R}(0)\|^-_{3,\times}|s|^{-3}$$

and hence $\Psi \in \mathcal{X}^-_{2,4\gamma,4\rho_1} \times \mathcal{X}^-_{2,4\gamma,4\rho_1}$. Finally, following the same procedure we get that $\Psi \in \mathcal{X}^-_{3,\gamma_0,\rho_0} \times \mathcal{X}^-_{3,\gamma_0,\rho_0}$ for some $\gamma_0 \geq \gamma$ and $\rho_0 \geq \rho$.

It is not difficult to see that, following the computations performed in the previous lemma, if $\Psi \in \mathcal{X}^-_{3,\gamma_0,\rho_0}$, then $\mathcal{R}(\Psi) - \mathcal{R}(0) \in \mathcal{X}^-_{5,\gamma_0,\rho_0}$ and henceforth, $\mathcal{B}^-(\mathcal{R}(\Psi)-$

$\mathcal{R}(0)) \in \mathcal{X}^-_{4,\gamma_0,\rho_0}$. Therefore, taking ρ_0 big enough, $\|\mathcal{B}^-(\mathcal{R}(\Psi)-\mathcal{R}(0))\|^-_{3,\times} < R/8$ and this implies that, by definition of R:

$$\|\Psi\|^-_{3,\times} \leq \|\mathcal{B}^-(\mathcal{R}(0))\|^-_{3,\times} + \|\mathcal{B}^-(\mathcal{R}(\Psi)-\mathcal{R}(0))\|^-_{3,\times} \leq \frac{R}{8}+\frac{R}{8}=\frac{R}{4}.$$

Thus, Ψ is Φ^-, the solution found by the fixed point theorem, and we are done.

This ends the proof of Proposition 2.3. □

3. **Asymptotic expression for the difference $\Phi^- - \Phi^+$.** When we will not want to stress the definition domain, we will omit the dependence on γ,ρ of the Banach space $\mathcal{Y}_{\nu,\gamma,\rho}$ so we will write $\mathcal{Y}_\nu$ instead of $\mathcal{Y}_{\nu,\gamma,\rho}$. If there is no danger of confusion, the dependence on ε will be also omitted.

Let $\Phi^\pm$ the solutions given by Proposition 2.3. We define $\Delta\phi = \phi^- - \phi^+$, $\Delta\varphi = \varphi^- - \varphi^+$ and $\Delta\Phi = \Phi^- - \Phi^+$ and we note that $\Delta\Phi$ is defined on $E_{\gamma,\rho} \subset D^-_{\gamma,\rho} \cap D^+_{\gamma,\rho}$ and it belongs to $\mathcal{Y}_3 \times \mathcal{Y}_3$.

From now on we suppose that $\alpha > 0$. The case $\alpha < 0$ is analogous.

Our goal in this section is to finish the proof of Theorem 1.4 and to provide a proof of Theorem 1.2. The main part of this section is to check the following proposition, which deals with the asymptotic expression of $\Delta\Phi$.

Proposition 3.1. *Let $\alpha > 0$. Given $\gamma > 0$ there exist $\rho > 0$ big enough, $C = (\kappa, 0) \in \mathbb{C}^2$ and $\chi \in \mathcal{Y}_1 \times \mathcal{Y}_2$ such that*

$$\Delta\Phi(s) = s\,\mathrm{e}^{-\mathrm{i}(\alpha s+\beta(s,\varepsilon))}\,\varepsilon(C+\xi(s)), \qquad s \in E_{\gamma,\rho}.$$

where $\beta(s,\varepsilon) = -(c+\varepsilon\alpha h_0)\log\big(s(1+\varepsilon h_0 s^{-1})\big)$ is defined in Lemma 2.1. Moreover $C \neq 0$ if and only if $\Delta\Phi \neq 0$.

The main idea of the proof of Proposition 3.1 is to study the differential equation that $\Delta\Phi$ verifies and to use that $\Delta\Phi$ is a bounded and analytic function on a sector of the lower complex half plane, $E_{\gamma,\rho}$. From these properties of $\Delta\Phi$ we will deduce its asymptotic behavior in $E_{\gamma,\rho}$.

Subtracting equations (2.3) for Φ^- and Φ^+ we get that $\Delta\Phi$ must satisfy the differential equation

$$\Delta\Phi' = \big[(1+\varepsilon h_0 s^{-1})^{-1}A(s) + R(s)\big]\Delta\Phi \tag{3.1}$$

where R is the matrix defined by

$$R(s) = \begin{pmatrix} R_1(s)^T \\ R_2(s)^T \end{pmatrix} = \int_0^1 D\mathcal{R}(\Phi^+(s)+\lambda(\Phi^-(s)-\Phi^+(s)))\,d\lambda$$

and $\mathcal{R}$ was given by (2.2). Since $\Delta\Phi$ satisfies equation (3.1), there exists a constant $K = (\kappa_0, \kappa_1) \in \mathbb{C}^2$ such that

$$\Delta\Phi(s) = M(s)\left[K + \int_{-\mathrm{i}\rho}^s M(t)^{-1}R(t)\Delta\Phi(t)\,dt\right]$$

where M was given in Lemma 2.1. Using the expression (2.5) of M, we have that

$$\Delta\phi(s) = s(1+\varepsilon h_0 s^{-1})\,\mathrm{e}^{-\mathrm{i}(\alpha s+\beta(s,\varepsilon))}\left[\kappa_0 + \int_{-\mathrm{i}\rho}^s \frac{\mathrm{e}^{\mathrm{i}(\alpha t+\beta(t,\varepsilon))}}{t(1+\varepsilon h_0 t^{-1})}\langle R_1(t), \Delta\Phi(t)\rangle\,dt\right]$$

$$\Delta\varphi(s) = s(1+\varepsilon h_0 s^{-1})\,\mathrm{e}^{\mathrm{i}(\alpha s+\beta(s,\varepsilon))}\left[\kappa_1 + \int_{-\mathrm{i}\rho}^s \frac{\mathrm{e}^{-\mathrm{i}(\alpha t+\beta(t,\varepsilon))}}{t(1+\varepsilon h_0 t^{-1})}\langle R_2(t), \Delta\Phi(t)\rangle\,dt\right]$$

where $\langle\cdot\rangle$ denotes the escalar product. We notice that, since $\Delta\Phi \in \mathcal{Y}_3 \times \mathcal{Y}_3$, we have that $\lim_{\mathrm{Im}\, s\to -\infty} \Delta\Phi(s) = 0$. On the other hand, $s\,\mathrm{e}^{\mathrm{i}(\alpha s+\beta(s,\varepsilon))}$ is not bounded as $\mathrm{Im}\, s \to -\infty$. So that we can deduce that $\kappa_1 = -\int_{-\mathrm{i}\rho}^{-\mathrm{i}\infty} \mathrm{e}^{-\mathrm{i}(\alpha t+\beta(t,\varepsilon))} t^{-1}(1 + \varepsilon h_0 t^{-1})^{-1}\langle R_2(t), \Delta\Phi(t)\rangle\, dt$ and therefore

$$\begin{aligned}\Delta\phi(s) =& s(1+\varepsilon h_0 s^{-1})\,\mathrm{e}^{-\mathrm{i}(\alpha s+\beta(s,\varepsilon))}\left[\kappa_0 + \int_{-\mathrm{i}\rho}^{s} \frac{\mathrm{e}^{\mathrm{i}(\alpha t+\beta(t,\varepsilon))}}{t(1+\varepsilon h_0 t^{-1})}\langle R_1(t), \Delta\Phi(t)\rangle\, dt\right]\\ \Delta\varphi(s) =& s(1+\varepsilon h_0 s^{-1})\,\mathrm{e}^{\mathrm{i}(\alpha s+\beta(s,\varepsilon))}\int_{-\mathrm{i}\infty}^{s} \frac{\mathrm{e}^{-\mathrm{i}(\alpha t+\beta(t,\varepsilon))}}{t(1+\varepsilon h_0 t^{-1})}\langle R_2(t), \Delta\Phi(t)\rangle\, dt. \end{aligned}\tag{3.2}$$

Once we know that $\Delta\Phi = (\Delta\phi, \Delta\varphi)$ satisfies the integral equation (3.2), we proceed to obtain an asymptotic expression for it.

Remark 3.2. *Using that $\Phi^{\pm} \in \mathcal{Y}_3\times\mathcal{Y}_3$ and definition (2.2) of $\mathcal{R}$, it is not difficult to check that all the coefficients of matrix R belong to $\mathcal{Y}_2$, hence $R_1, R_2 \in \mathcal{Y}_2\times\mathcal{Y}_2$.*

We define the linear operator $\mathcal{G}$ by the expression:

$$\mathcal{G}(\Psi)(s) = s(1+\varepsilon h_0 s^{-1})\begin{pmatrix} \mathrm{e}^{-\mathrm{i}(\alpha s+\beta(s,\varepsilon))}\displaystyle\int_{-\mathrm{i}\rho}^{s} \frac{\mathrm{e}^{\mathrm{i}(\alpha t+\beta(t,\varepsilon))}}{t(1+\varepsilon h_0 t^{-1})}\langle R_1(t), \Psi(t)\rangle\, dt \\ \mathrm{e}^{\mathrm{i}(\alpha s+\beta(s,\varepsilon))}\displaystyle\int_{-\mathrm{i}\infty}^{s} \frac{\mathrm{e}^{-\mathrm{i}(\alpha t+\beta(t,\varepsilon))}}{t(1+\varepsilon h_0 t^{-1})}\langle R_2(t), \Psi(t)\rangle\, dt \end{pmatrix}$$

and the function

$$\Delta\Phi_0(s) = s(1+\varepsilon h_0 s^{-1})\begin{pmatrix} \kappa_0\,\mathrm{e}^{-\mathrm{i}(\alpha s+\beta(s,\varepsilon))} \\ 0 \end{pmatrix}. \tag{3.3}$$

We observe that

$$\Delta\Phi(s) = \Delta\Phi_0(s) + \mathcal{G}(\Delta\Phi)(s). \tag{3.4}$$

Our strategy to prove the result will be to check that $\Delta\Phi = (\mathrm{Id}-\mathcal{G})^{-1}(\Delta\Phi_0)$. For that, since we only know that $\Delta\Phi \in \mathcal{Y}_3\times\mathcal{Y}_3$, we will prove that the linear operator $\mathrm{Id}-\mathcal{G}$ is invertible on $\mathcal{Y}_3\times\mathcal{Y}_3$. After that we will study how the operator $(\mathrm{Id}-\mathcal{G})^{-1}$ acts on $\Delta\Phi_0$.

We introduce the auxiliary linear operator $\mathcal{L}^{\alpha}$ defined for $h = (h_1, h_2)$ as:

$$\mathcal{L}^{\alpha}(h)(s) = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}(\alpha s+\beta(s,\varepsilon))}\displaystyle\int_{-\mathrm{i}\rho}^{s} \mathrm{e}^{\mathrm{i}(\alpha t+\beta(t,\varepsilon))}\, h_1(t)\, dt \\ \mathrm{e}^{\mathrm{i}(\alpha s+\beta(s,\varepsilon))}\displaystyle\int_{-\mathrm{i}\infty}^{s} \mathrm{e}^{-\mathrm{i}(\alpha t+\beta(t,\varepsilon))}\, h_2(t)\, dt \end{pmatrix}. \tag{3.5}$$

Lemma 3.3. *For any $\nu \geq 0$, $\alpha > 0$ and $\gamma > 0$, there exists $\rho > 0$ big enough such that $\mathcal{L}^{\alpha} : \mathcal{Y}_\nu\times\mathcal{Y}_\nu \to \mathcal{Y}_\nu\times\mathcal{Y}_\nu$. Moreover, there exists a constant $C > 0$ such that $\|\mathcal{L}^{\alpha}(h)\|_{\nu,\times} \leq C\|h\|_{\nu,\times}$.*

Proof. We note that there exists a constant K (depending on $\gamma, \alpha, c, \varepsilon_0$) such that

$$|\,\mathrm{e}^{\mathrm{i}(\beta(s,\varepsilon)-\beta(t,\varepsilon))}\,| \leq K, \qquad \text{for all } s, t \in E_{\gamma,\rho}. \tag{3.6}$$

To prove this bound we proceed in a similar way as in (2.11).

Let $h = (h_1, h_2) \in \mathcal{Y}_\nu\times\mathcal{Y}_\nu$. We write $\mathcal{L}^{\alpha}(h) = (\mathcal{L}_1^{\alpha}(h_1), \mathcal{L}_2^{\alpha}(h_2))$. First we deal with $\mathcal{L}_2^{\alpha}(h_2)$. By Cauchy's theorem,

$$\mathcal{L}_2^{\alpha}(h_2)(s) = \mathrm{i}\int_{-\infty}^{0} \mathrm{e}^{\alpha t}\,\mathrm{e}^{\mathrm{i}(\beta(s,\varepsilon)-\beta(s+\mathrm{i}\,t,\varepsilon))}\, h_2(s+\mathrm{i}\,t)\, dt$$

Therefore, using (3.6) and that $h_2 \in \mathcal{Y}_\nu$,

$$|\mathcal{L}_2^\alpha(h_2)(s)| \le K\|h_2\|_\nu \int_{-\infty}^0 \mathrm{e}^{\alpha t}\,\frac{1}{|s+\mathrm{i}\,t|^\nu}\,dt \le K\|h\|_{\nu,\times}|s|^{-\nu}\alpha^{-1}. \tag{3.7}$$

Now we deal with $\mathcal{L}_1^\alpha(h_1)$. Let $c_\gamma = \gamma^{-1}(1+\gamma^2)$. We notice that, using again bound (3.6),

$$\begin{aligned}
|\mathcal{L}_1^\alpha(h_1)(s)| &\le K\Big|\int_{-\mathrm{i}\,\rho}^s \big|\mathrm{e}^{\mathrm{i}\,\alpha(t-s)}\,h_1(t)\big|\,dt\Big| \\
&\le K\frac{|s+\mathrm{i}\,\rho|}{|\operatorname{Im} s+\rho|}\|h_1\|_\nu \int_0^{|\operatorname{Im} s+\rho|}\frac{\mathrm{e}^{-\alpha t}}{|t+\operatorname{Im} s|^\nu}\,dt \\
&\le Kc_\gamma\|h\|_{\nu,\times}\int_0^{|\operatorname{Im} s+\rho|}\frac{\mathrm{e}^{-\alpha t}}{|t+\operatorname{Im} s|^\nu}\,dt
\end{aligned} \tag{3.8}$$

provided that $|s+\mathrm{i}\,\rho||\operatorname{Im} s+\mathrm{i}\,\rho|^{-1} < c_\gamma$. We define $I_\nu = \int_0^{|\operatorname{Im} s+\rho|}\mathrm{e}^{-\alpha t}\,|t+\operatorname{Im} s|^{-\nu}\,dt$. Integrating by parts I_ν, it is easily checked that $I_\nu \le \alpha^{-1}\big(|\operatorname{Im} s|^{-\nu}+\nu\rho^{-1}I_\nu\big)$ and therefore, if $\rho > 2\nu\alpha^{-1}$, $I_\nu \le 2\alpha^{-1}|\operatorname{Im} s|^{-\nu}$. Hence, bounding (3.8) and using that $|s| \le c_\gamma|\operatorname{Im} s|$, we get

$$|\mathcal{L}_1^\alpha(h_1)(s)| \le 2Kc_\gamma^{\nu+1}\alpha^{-1}|s|^{-\nu}\|h\|_{\nu,\times}$$

and Lemma 3.3 is proved. □

Lemma 3.4. *The map $\mathcal{G}:\mathcal{Y}_3\times\mathcal{Y}_3\to\mathcal{Y}_3\times\mathcal{Y}_3$ is well defined. Moreover, if ρ is big enough,* $\operatorname{Id}-\mathcal{G}$ *is invertible.*

Proof. We write $\mathcal{G}=(\mathcal{G}_1,\mathcal{G}_2)$ and we fix $s\in E_{\gamma,\rho}$ and $\Psi\in\mathcal{Y}_3\times\mathcal{Y}_3$. We define the auxiliary function $h=(h_1,h_2)$ by

$$h(t) = \frac{1}{t(1+\varepsilon h_0 t^{-1})}R(t)\Psi(t). \tag{3.9}$$

We observe that

$$\mathcal{G}(\Psi)(s) = s(1+\varepsilon h_0 s^{-1})\mathcal{L}^\alpha(h)(s). \tag{3.10}$$

We note that, since $R_1,R_2\in\mathcal{Y}_2\times\mathcal{Y}_2$ and Ψ belongs to $\mathcal{Y}_3\times\mathcal{Y}_3$, we have that $h\in\mathcal{Y}_6\times\mathcal{Y}_6$, if $\rho>\varepsilon_0 h_0$, and moreover

$$\|h\|_{6,\times} \le 3\max\{\|R_1\|_{2,\times},\|R_2\|_{2,\times}\}\|\Psi\|_{3,\times} \tag{3.11}$$

if ρ is big enough. By identity (3.10) and using Lemma 3.3 and (3.11) to bound $\mathcal{G}(\Psi)(s)$ we get:

$$\begin{aligned}
\|\mathcal{G}(\Psi)(s)\| &\le (1+\varepsilon_0|h_0||s|^{-1})3C|s|^{-5}\max\{\|R_1\|_{2,\times},\|R_2\|_{2,\times}\}\|\Psi\|_{3,\times} \\
&\le 6C|s|^{-5}\max\{\|R_1\|_{2,\times},\|R_2\|_{2,\times}\}\|\Psi\|_{3,\times}
\end{aligned}$$

if ρ is big enough. We recall that $\|\cdot\|$ denotes the supremum norm in $\mathbb{C}^2$. Hence $\mathcal{G}(\Psi)\in\mathcal{Y}_5\times\mathcal{Y}_5\subset\mathcal{Y}_3\times\mathcal{Y}_3$ and since $|s|>\rho$ if $s\in E_{\gamma,\rho}$, we obtain

$$\|\mathcal{G}(\Psi)\|_{3,\times} \le 6C\max\{\|R_1\|_{2,\times},\|R_2\|_{2,\times}\}\|\Psi\|_{3,\times}\rho^{-2} \le \frac{1}{2}\|\Psi\|_{3,\times} \tag{3.12}$$

taking ρ big enough. Since $\mathcal{G}$ is a linear operator, this implies that $\operatorname{Id}-\mathcal{G}$ is invertible. □

We recall that in (3.3) we had defined $\Delta\Phi_0$ as:

$$\Delta\Phi_0(s) = s(1+\varepsilon h_0 s^{-1})\,\mathrm{e}^{-\,\mathrm{i}(\alpha s+\beta(s,\varepsilon))}\begin{pmatrix}\kappa_0\\ 0\end{pmatrix}.$$

We observe that, by Lemma 3.4 and expression (3.4) of $\Delta\Phi$,

$$\Delta\Phi = (\mathrm{Id}-\mathcal{G})^{-1}(\Delta\Phi_0) = \sum_{n\geq 0}\mathcal{G}^n(\Delta\Phi_0). \tag{3.13}$$

Once we have proved that $\Delta\Phi$ can be obtained from formula (3.13), we have to see how the operator $\mathcal{G}$ and its iterates $\mathcal{G}^n$ act on $\Delta\Phi_0$. For that we introduce new Banach spaces. At the end, we will get exponentially small bounds for $\Delta\Phi$.

For any $\nu\geq 0$, we define the Banach space

$$\mathcal{Z}_\nu = \{h : E_{\gamma,\rho}\to C,\text{ analytic and } \|h\|_{\nu,\mathrm{e}} := \sup_{s\in E_{\gamma,\rho}} |s^\nu\,\mathrm{e}^{\mathrm{i}(\alpha s+\beta(s,\varepsilon))}\,h(s)| < +\infty\}.$$

We note that $\Delta\Phi_0\in\mathcal{Z}_{-1}\times\mathcal{Z}_l$ for all $l\geq 0$. For later convenience we are going to study how the operator $\mathcal{G}$ acts on functions of $\mathcal{Z}_{-1}\times\mathcal{Z}_0$. We define the norm on $\mathcal{Z}_{-1}\times\mathcal{Z}_0$

$$\|(h_1,h_2)\|_{-1,0} = \max\{\|h_1\|_{-1,\mathrm{e}},\|h_2\|_{0,\mathrm{e}}\}.$$

Lemma 3.5. *For any $\gamma>0$, there exists ρ big enough such that the operator $\mathcal{G}:\mathcal{Z}_{-1}\times\mathcal{Z}_0\to\mathcal{Z}_{-1}\times\mathcal{Z}_1$ is well defined. In addition, for any $h\in\mathcal{Z}_{-1}\times\mathcal{Z}_0$, there exists a constant $\bar K(\rho)$ depending on ρ such that*

$$\pi^1\mathcal{G}(h) - s\,\mathrm{e}^{-\,\mathrm{i}(\alpha s+\beta(s,\varepsilon))}\,\bar K(\rho)\in\mathcal{Z}_0.$$

Moreover the linear operator $Id-\mathcal{G}$ is invertible on $\mathcal{Z}_{-1}\times\mathcal{Z}_0$.

Proof. Let $h=(h_1,h_2)\in\mathcal{Z}_{-1}\times\mathcal{Z}_0$. We introduce the auxiliary function

$$\bar h(t) = (\bar h_1(t),\bar h_2(t)) = \mathrm{e}^{\mathrm{i}(\alpha t+\beta(t,\varepsilon))}\,\frac{1}{t(1+\varepsilon h_0 t^{-1})}R(t)h(t)$$

and we claim that $\bar h\in\mathcal{Y}_2\times\mathcal{Y}_2$ for $\rho>\varepsilon_0 h_0$. Indeed, let $s\in E_{\gamma,\rho}$, then, by definition of $\mathcal{Z}_{-1}\times\mathcal{Z}_0$,

$$h_1\,\mathrm{e}^{\mathrm{i}(\alpha t+\beta(t,\varepsilon))}\in\mathcal{Y}_{-1},\qquad h_2\,\mathrm{e}^{\mathrm{i}(\alpha t+\beta(t,\varepsilon))}\in\mathcal{Y}_0$$

and the claim is proved provided that $R_1,R_2\in\mathcal{Y}_2\times\mathcal{Y}_2$. Moreover it is not difficult to check that, if ρ is big enough

$$\|\bar h\|_{2,\times}\leq 2\max\{\|R_1\|_{2,\times},\|R_2\|_{2,\times}\}\|h\|_{-1,0}. \tag{3.14}$$

We write $\mathcal{G}(h)=(\mathcal{G}_1(h),\mathcal{G}_2(h))$. For any $s\in E_{\gamma,\rho}$,

$$\mathcal{G}_1(h)(s) = \mathrm{e}^{-\,\mathrm{i}(\alpha s+\beta(s,\varepsilon))}\,s(1+\varepsilon h_0 s^{-1})\int_{-\,\mathrm{i}\,\rho}^{s}\bar h_1(t)\,dt$$
$$\mathcal{G}_2(h)(s) = \mathrm{e}^{\mathrm{i}(\alpha s+\beta(s,\varepsilon))}\,s(1+\varepsilon h_0 s^{-1})\int_{-\,\mathrm{i}\,\infty}^{s}\mathrm{e}^{-2\,\mathrm{i}(\alpha t+\beta(t,\varepsilon))}\,\bar h_2(t)\,dt$$

Now we deal with $\mathcal{G}_2(h)$. We observe that

$$\frac{\mathrm{e}^{\mathrm{i}(\alpha s+\beta(s,\varepsilon))}}{s(1+\varepsilon h_0 s^{-1})}\mathcal{G}_2(h)(s) = \mathrm{e}^{2\,\mathrm{i}(\alpha s+\beta(s,\varepsilon))}\int_{-\,\mathrm{i}\,\infty}^{s}\mathrm{e}^{-2\,\mathrm{i}(\alpha t+\beta(t,\varepsilon))}\,\bar h_2(t) = \mathcal{L}_2^{2\alpha}(\bar h_2)(s)$$

where $\mathcal{L}^{2\alpha}$ was defined in (3.5). Taking into account that $\bar h_2\in\mathcal{Y}_2$ and Lemma 3.3, we conclude that $\mathrm{e}^{\mathrm{i}(\alpha s+\beta(s,\varepsilon))}\,s^{-1}\mathcal{G}_2(h)(s)\in\mathcal{Y}_2$ and henceforth $\mathcal{G}_2(h)(s)\in\mathcal{Z}_1$.

Moreover, using again Lemma 3.3 and (3.14) to bound $\|\bar h\|_{2,\times}$, we obtain that for all $s\in E_{\gamma,\rho}$,

$$\begin{aligned}|\,\mathrm{e}^{\mathrm{i}(\alpha s+\beta(s,\varepsilon))}\,\mathcal{G}_2(h)(s)| &= |s(1+\varepsilon h_0 s^{-1})\mathcal{L}_2^{2\alpha}(\bar h_2)(s)| \le 2C|s|^{-1}\|\bar h\|_{2,\times}\\ &\le 2C\rho^{-1}\|\bar h\|_{2,\times} \le 4C\rho^{-1}\max\{\|R_1\|_{2,\times},\|R_2\|_{2,\times}\}\|h\|_{-1,0}\\ &\le \frac{1}{4}\|h\|_{-1,0} \qquad (3.15)\end{aligned}$$

if ρ is big enough.

Now we deal with $\mathcal{G}_1(h)$. First we note that, since $\bar h_1\in\mathcal{Y}_2$, by Cauchy's theorem,

$$\int_{-\mathrm{i}\rho}^{s}\bar h_1(t)\,dt=\int_{-\mathrm{i}\rho}^{-\mathrm{i}\infty}\bar h_1(t)\,dt+\int_{-\mathrm{i}\infty}^{s}\bar h_1(t)\,dt. \qquad (3.16)$$

It is straightforward to check that, if $u\in E_{\gamma,\rho}$,

$$\begin{aligned}\left|\int_{-\mathrm{i}\infty}^{u}\bar h_1(t)\,dt\right| &= \int_{-\infty}^{0}|\bar h_1(u+\mathrm{i}\,t)|\,dt \le \|\bar h_1\|_2\int_{-\infty}^{0}|u+\mathrm{i}\,t|^{-2}\,dt\\ &\le \|\bar h_1\|_2\int_{-\infty}^{0}(|u|^2+t^2)^{-1}\,dt=\|\bar h_1\|_2\frac{\pi}{2}|u|^{-1}. \qquad (3.17)\end{aligned}$$

Hence the operator $\mathcal{G}_1:\mathcal{Z}_{-1}\times\mathcal{Z}_0\to\mathcal{Z}_{-1}$ is well defined. Moreover, by decomposition (3.16) and estimate (3.17), we have that

$$\mathcal{G}_1(h)-s(1+\varepsilon h_0 s^{-1})\,\mathrm{e}^{-\mathrm{i}(\alpha s+\beta(s,\varepsilon))}\int_{-\mathrm{i}\rho}^{-\mathrm{i}\infty}\bar h_1(t)\,dt\in\mathcal{Z}_0.$$

Using decomposition (3.16) and (3.14) to bound $\|\bar h_1\|_2$, we have that

$$\left|\int_{-\mathrm{i}\rho}^{s}\bar h_1(t)\,dt\right|\le\|\bar h_1\|_2\frac{\pi}{2}(\rho^{-1}+|s|^{-1})\le\max\{\|R_1\|_{2,\times},\|R_2\|_{2,\times}\}\|h\|_{-1,0}\pi\rho^{-1}$$

and we obtain that, if ρ is big enough

$$|\mathcal{G}_1(h)(s)s^{-1}\,\mathrm{e}^{\mathrm{i}(\alpha+\beta(s,\varepsilon))}\,|\le 2\max\{\|R_1\|_{2,\times},\|R_2\|_{2,\times}\}\|h\|_{-1,0}\pi\rho^{-1}.$$

Finally we conclude that, taking if necessary ρ big enough,

$$\|\mathcal{G}_1(h)\|_{-1,\mathrm{e}}\le\frac{1}{4}\|h\|_{-1,0} \qquad (3.18)$$

and the lemma is proved. □

Now we are going to finish the proof of the main proposition in this section.

End of the proof of Proposition 3.1. Proposition 3.1 follows, mainly, from the fact that $\Delta\Phi_0\in\mathcal{Z}_{-1}\times\mathcal{Z}_0$, $\mathrm{Id}-\mathcal{G}$ is invertible on this Banach space and hence

$$\Delta\Phi=(\mathrm{Id}-\mathcal{G})^{-1}(\Delta\Phi_0)\in\mathcal{Z}_{-1}\times\mathcal{Z}_0. \qquad (3.19)$$

We notice that, by Lemma 3.5, $\mathcal{G}(\Delta\Phi)\in\mathcal{Z}_{-1}\times\mathcal{Z}_1$ provided $\Delta\Phi\in\mathcal{Z}_{-1}\times\mathcal{Z}_0$. Then, since $\Delta\Phi=\Delta\Phi_0+\mathcal{G}(\Delta\Phi)$, we have obtained that $\Delta\Phi\in\mathcal{Z}_{-1}\times\mathcal{Z}_1$. In addition, again by Lemma 3.5, there exists a constant $\bar K(\rho)$ such that

$$\pi^1(\Delta\Phi-\Delta\Phi_0)-s\,\mathrm{e}^{-\mathrm{i}(\alpha s+\beta(s,\varepsilon))}\,\bar K(\rho)\in\mathcal{Z}_0$$

and therefore

$$\Delta\Phi=s\,\mathrm{e}^{-\mathrm{i}(\alpha s+\beta(s,\varepsilon))}(K+\chi(s))$$

where $\chi\in\mathcal{Y}_1\times\mathcal{Y}_2$, $K\in\mathbb{C}^2$, $\pi^2K=0$ and $\pi^1K=\kappa_0+\bar K(\rho)$. We observe that, since $\Delta\Phi$ is independent of ρ, the constant K is independent of ρ too.

Now we prove that $K \neq 0$ if and only if $\Delta\Phi \neq 0$. If there exists $s_0 \in E_{\gamma,\rho}$ such that $\Delta\Phi(s_0) = 0$, then $\Delta\Phi(s) = 0$ for all $s \in E_{\gamma,\rho}$ (this fact is obvious since $\Delta\Phi$ is a solution of an homogeneous linear equation), hence in this case, clearly, $K = 0$.

On the other hand, if $\Delta\Phi(s) \neq 0$ for all $s \in E_{\gamma,\rho}$, then $\kappa_0 \neq 0$ (on the contrary, $\Delta\Phi_0 = 0$ and by equality (3.19), $\Delta\Phi = 0$). We recall that, from (3.18) and (3.15), we have that $\|\mathcal{G}\|_{-1,0} \leq 1/4$. Henceforth, since

$$\Delta\Phi - \Delta\Phi_0 = \sum_{n\geq 1} \mathcal{G}^n(\Delta\Phi_0)$$

we have that

$$\|\Delta\Phi - \Delta\Phi_0\|_{-1,0} \leq \sum_{n\geq 1} \frac{1}{4^n}\|\Delta\Phi_0\|_{-1,0} = \frac{1}{3}\kappa_0.$$

By definition of the norm $\|\cdot\|_{-1,0}$, the last bound implies that for any $s \in E_{\gamma,\rho}$,

$$|\pi^1\Delta\Phi(s)s^{-1}\,\mathrm{e}^{\mathrm{i}(\alpha s+\beta(s,\varepsilon))}| \geq |\pi^1\Delta\Phi_0(s)s^{-1}\,\mathrm{e}^{\mathrm{i}(\alpha s+\beta(s,\varepsilon))}| - \frac{1}{3}\kappa_0,$$

hence, using that $\pi^1\Delta\Phi_0(s)s^{-1}\,\mathrm{e}^{\mathrm{i}(\alpha s+\beta(s,\varepsilon))} = \kappa_0$, we have that for all $s \in E_{\gamma,\rho}$,

$$|\pi^1\Delta\Phi(s)s^{-1}\,\mathrm{e}^{\mathrm{i}(\alpha s+\beta(s,\varepsilon))}| = |\pi^1 K + \pi^1\chi(s)| \geq \frac{2}{3}|\kappa_0|$$

and henceforth, taking $\operatorname{Im} s \to -\infty$, we get that $|\pi^1 K| \geq \frac{2}{3}|\kappa_0|$ and thus $K \neq 0$.

In addition, by Proposition 2.3, we have that $\Phi^\pm(s,\varepsilon) = \Phi_0^\pm(s,\varepsilon) + \Phi_1^\pm(s,\varepsilon)$ with $\Phi_0^\pm(s,\varepsilon) = \mathcal{B}^\pm \circ \mathcal{R}(0)(s,\varepsilon)$, where $\mathcal{R}(\Phi)(s,\varepsilon)$ is given in (2.16), and $\Phi_1^\pm(s,\varepsilon)$ satisfying $\|\Phi_1^\pm\|_{3,\times} < \|\Phi_0^\pm\|_{3,\times}$. As $\mathcal{R}(0)(s,0) = 0$, it is clear that $\Phi_0^\pm(s,0) = 0$ and consequently $\Phi^\pm(s,0) = 0$. Thus, if $\varepsilon = 0$, then $\Delta\Phi = 0$ and this implies that $K = 0$ and henceforth $\chi = 0$. We conclude then that $K = \varepsilon C$ for some constant C (depending on ε) and $\chi = \varepsilon\xi$ with $\xi \in \mathcal{Y}_1 \times \mathcal{Y}_2$. □

3.1. **The case ε small.** In this subsection we are going to prove the results in Theorem 1.4 related with the value of the constant C at $\varepsilon = 0$.

Since system (2.3) is analytic with respect to ε, it is clear that the solutions $\Phi^\pm$ can be expressed of the form

$$\Phi^\pm(s,\varepsilon) = \Phi^\pm(s,0) + \varepsilon\partial_\varepsilon\Phi^\pm(s,0) + \varepsilon^2\overline{\Phi}^\pm(s,\varepsilon).$$

On the other hand, as we have pointed out before, $\Phi^\pm(s,0) = 0$. Hence the variational equation for $\partial_\varepsilon\Phi^\pm(s,0)$ is given by

$$\frac{d}{ds}(\partial_\varepsilon\Phi^\pm(s,0)) = A(s)\partial_\varepsilon\Phi^\pm(s,0) + \partial_\varepsilon\mathcal{R}(0)(s,0)$$

and since $\varepsilon\partial_\varepsilon\mathcal{R}(0)(s,0) + O(\varepsilon^2) = \mathcal{R}(0)(s,\varepsilon)$, and $\partial_\varepsilon\Phi^\pm(s,0)$ goes to 0 as $\operatorname{Re} s \to \pm\infty$ respectively, we have that

$$\varepsilon\partial_\varepsilon\Phi^\pm(s,0) = \mathcal{B}^\pm(\varepsilon\partial_\varepsilon\mathcal{R}(0))(s,0) = \mathcal{B}^\pm(\mathcal{R}(0))(s,\varepsilon) + O(\varepsilon^2) = \Phi_0^\pm(s,\varepsilon) + O(\varepsilon^2).$$

Therefore, by definition of $\mathcal{B}^\pm$ and $\mathcal{R}$ we have that

$$\begin{aligned}\Delta\Phi(s,\varepsilon) &= \varepsilon(\partial_\varepsilon\Phi^-(s,0) - \partial_\varepsilon\Phi^+(s,0)) + O(\varepsilon^2)\\ &= (\mathcal{B}^- \circ \mathcal{R}(0)(s,\varepsilon) - \mathcal{B}^+ \circ \mathcal{R}(0)(s,\varepsilon)) + O(\varepsilon^2)\\ &= \varepsilon M(s)\int_{-\infty}^{+\infty} M(s+t)^{-1}F(0,-(s+t)^{-1})\,dt + O(\varepsilon^2).\end{aligned}$$

(We stress that $M(s)$ and $M(s+t)$ are evaluated at $\varepsilon=0$). Finally from the above equality and Theorem 1.4 one deduces that

$$\begin{aligned} s\,\mathrm{e}^{-\,\mathrm{i}(\alpha s+\beta(s,0))}(C(0)+\xi(s,0)) &= M(s)\int_{-\infty}^{+\infty} M(s+t)^{-1}F(0,-(s+t)^{-1})\,dt \\ &= s\left(\begin{array}{c} \mathrm{e}^{\mathrm{i}\,c\log s}\displaystyle\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}(\alpha t-c\log(s+t))}\,\frac{1}{s+t}F_1(0,-(s+t)^{-1})\,dt \\ \mathrm{e}^{-\,\mathrm{i}\,c\log s}\displaystyle\int_{-\infty}^{+\infty}\mathrm{e}^{-\,\mathrm{i}(\alpha t-c\log(s+t))}\,\frac{1}{s+t}F_2(0,-(s+t)^{-1})\,dt \end{array}\right) \\ &:= s\left(\begin{array}{c}\mathcal{M}_1(s)\\ \mathcal{M}_2(s)\end{array}\right). \end{aligned} \tag{3.20}$$

Now we are going to estimate $\mathcal{M}_1$ and $\mathcal{M}_2$. For that we observe that

$$F_1(0,-s^{-1})=\sum_{n\geq 3}a_n(-1)^n s^{-n},\qquad F_2(0,-s^{-1})=\sum_{n\geq 3}b_n(-1)^n s^{-n}.$$

Hence we have that

$$\mathcal{M}_1(s)=\sum_{n\geq 3}a_n(-1)^n\,\mathrm{e}^{\mathrm{i}\,c\log s}\int_{-\infty}^{+\infty}\frac{\mathrm{e}^{\mathrm{i}\,\alpha t}}{(s+t)^{n+1+\mathrm{i}\,c}}\,dt := \sum_{n\geq 3}a_n(-1)^n\,\mathrm{e}^{\mathrm{i}\,c\log s}\,I_n$$

$$\mathcal{M}_2(s)=\sum_{n\geq 3}b_n(-1)^n\,\mathrm{e}^{-\,\mathrm{i}\,c\log s}\int_{-\infty}^{+\infty}\frac{\mathrm{e}^{-\,\mathrm{i}\,\alpha t}}{(s+t)^{n+1-\mathrm{i}\,c}}\,dt := \sum_{n\geq 3}b_n(-1)^n\,\mathrm{e}^{-\,\mathrm{i}\,c\log s}\,J_n$$

In [Bal] was shown that

$$\begin{aligned} I_n &= \mathrm{i}^{n+1+\mathrm{i}\,c}\,\alpha^{n+\mathrm{i}\,c}\frac{2\pi}{\Gamma(n+1+\mathrm{i}\,c)}\,\mathrm{e}^{-\,\mathrm{i}\,\alpha s}(1+O(|\operatorname{Im}s|^{-1})) \\ |J_n| &\leq C\,\mathrm{e}^{-2\alpha|\operatorname{Im}s|}\,|s|^{-n} \end{aligned} \tag{3.21}$$

In fact, in [Bal] the asymptotic expression for I_n is proved only when I_n has the form $I_n=\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}\,\alpha t}(s+t)^{-\ell-1}\,dt$ with $\ell\in\mathbb{Q}$, but it is immediate that the result also holds in this case. The estimation for J_n also needs an extra argument to be done from the results in [Bal].

Using the asymptotic expressions in (3.21) we obtain that

$$\begin{aligned} \mathcal{M}_1(s) &= \sum_{n\geq 3}a_n(-1)^n\frac{2\pi\,\mathrm{i}}{\Gamma(n+1+\mathrm{i}\,c)}(\mathrm{i}\,\alpha)^{n+\mathrm{i}\,c}\,\mathrm{e}^{-\,\mathrm{i}(\alpha s-c\log s)}(1+O(|\operatorname{Im}s|)^{-1}) \\ \mathcal{M}_2(s) &= O(\mathrm{e}^{-2\,\mathrm{i}\,\alpha s}\,|s|^{-3}). \end{aligned}$$

Hence we have that $\pi^2 C(0)=0$. Finally, we define $m_1(u)=u^{-1-\mathrm{i}\,c}F_1(0,-u^{-1})=\sum_{n\geq 3}m^1_n u^{-n-1-\mathrm{i}\,c}$ and its Borel transform $\hat{m}_1(\zeta)=\sum_{n\geq 3}m^1_n\frac{\zeta^{n+\mathrm{i}\,c}}{\Gamma(n+1+\mathrm{i}\,c)}$ and hence, since $m^1_n=(-1)^n a_n$,

$$\mathcal{M}_1(s)=2\pi\,\mathrm{i}\,\hat{m}_1(\mathrm{i}\,\alpha)\,\mathrm{e}^{-\,\mathrm{i}(\alpha s-c\log s)}(1+O(|\operatorname{Im}s|)^{-1})$$

and by (3.20), this implies that

$$\pi^1 C(0)=2\pi\,\mathrm{i}\,\hat{m}_1(\mathrm{i}\,\alpha).$$

3.2. **Proof of Theorem 1.2.** In this section we will recover Theorem 1.2 from Theorem 1.4. We will need a technical lemma, analogous to Lemma 2.6, which was proved in [Bal].

Lemma 3.6. *Let $\nu, \rho, \gamma > 0$, and $h \in \mathcal{Y}_{\nu,\gamma,\rho}$. Then there exists a constant C such that for $l \in \mathbb{N}\backslash\{0\}$ we have that*

$$\partial_\tau^l h \in \mathcal{Y}_{l+\nu,2\gamma,2\rho} \qquad \text{and} \qquad \|\partial_\tau^l h\|_{l+\nu} \leq l! 2^{-l} C \|h\|_\nu.$$

Given $\Phi^\pm(s,\varepsilon) = (\phi^\pm(s,\varepsilon), \varphi^\pm(s,\varepsilon))$ the solutions obtained in Theorem 1.4, we consider the autonomous differential equations given by

$$\frac{ds}{d\tau} = 1 + s^2(b\phi^\pm(s,\varepsilon)\varphi^\pm(s,\varepsilon) + \varepsilon H(\phi^\pm(s,\varepsilon), \varphi^\pm(s,\varepsilon), -s^{-1})). \tag{3.22}$$

We fix $\gamma > 0$ and we take any $\rho_0 > 0$ such that the conclusions of Theorem 1.4 become true. Since $\phi^\pm, \varphi^\pm \in \mathcal{X}^\pm_{3,\gamma/4,\rho_0}$ and H can be decomposed of the form (2.1), equation (3.22) can be expressed as

$$\frac{ds}{d\tau} = 1 + \varepsilon h_0 s^{-1} + S^\pm(s,\varepsilon), \qquad S^\pm \in \mathcal{X}^\pm_{2,\gamma/4,\rho_0} \tag{3.23}$$

where $S^\pm$ is given by:

$$S^\pm(s) = \bar{H}(\phi^\pm(s), \varphi^\pm(s), -s^{-1}).$$

Lemma 3.7. *For any $\gamma > 0$, there exists $\rho \geq \rho_0$ big enough such that equation (3.23) has two solutions $s^\pm$ satisfying*

$$s^\pm(\tau,\varepsilon) = \tau + \varepsilon h_0 \log\tau + (\varepsilon h_0)^2\tau^{-1}\log\tau + \mathcal{S}^\pm(\tau,\varepsilon), \qquad \mathcal{S}^\pm \in \mathcal{X}^\pm_{1,\gamma,\rho}. \tag{3.24}$$

Let $\Delta s(\tau,\varepsilon) = s^-(\tau,\varepsilon) - s^+(\tau,\varepsilon)$. Then

$$\sup_{\tau\in E_{2\gamma,2\rho}} |\Delta s(\tau,\varepsilon)\tau^{-1} \mathrm{e}^{\mathrm{i}(\alpha\tau - c\log\tau)}| < \infty.$$

Proof. We look for solutions of equation (3.23). We deal only in the $-$ case being the $+$ analogous. Along this proof we do not write the dependence on the parameter ε.

We recall that, as we pointed out in (3.23), $S^- \in \mathcal{X}^-_{2,\gamma/4,\rho_0}$. We are interested in solutions of (3.23) of the form $s = s_0 + s_1$ with $s_0(\tau) = \tau + \varepsilon h_0\log\tau + (\varepsilon h_0)^2\tau^{-1}\log\tau$. For technical reasons, first we perform the change of coordinates given by $s = u + h(u) := u + \varepsilon h_0 \log u + (\varepsilon h_0)^2 u^{-1}[\log u + 1]$ and we obtain a new system

$$u' = 1 + \mathcal{U}^-(u), \qquad \text{with} \qquad \mathcal{U}^-(u) \in \mathcal{X}^-_{2,\gamma/2,\rho/4} \tag{3.25}$$

provided ρ is big enough. Indeed, it is clear that

$$\begin{aligned} u' =& \frac{1}{1 + \varepsilon h_0 u^{-1} - (\varepsilon h_0)^2 u^{-2}\log u} \\ & \cdot \left(1 + \varepsilon h_0 (u + \varepsilon h_0 \log u + (\varepsilon h_0)^2[u^{-1}\log u + u^{-1}])^{-1} + S^-(u + h(u))\right) \\ =& \left(1 - \varepsilon h_0 u^{-1} + (\varepsilon h_0)^2 u^{-2}\log u + O(u^{-2})\right) \\ & \cdot \left(1 + \varepsilon h_0 u^{-1} - (\varepsilon h_0)^2 u^{-2}\log u + O(u^{-3}(\log u)^2) + S^-(u + h(u))\right) \\ :=& 1 + \mathcal{U}^-(u). \end{aligned}$$

As we pointed out as a comment below Lemma 2.6, there exists a constant $C_{\gamma,\rho}$ such that if $u \in D^-_{\gamma,\rho}$, then the open ball of radius $C_{\gamma,\rho}|u|$ and center u is contained in $D^-_{2\gamma,4\rho}$. Hence, taking ρ big enough so that $|h(u)| = |\varepsilon h_0 \log u + (\varepsilon h_0)^2 u^{-1}[\log u +$

$1]| \leq C_{\gamma,\rho}|u|$ we have that $u+h(u) \in D^-_{\gamma/4,\rho/16}$ for $u \in D^-_{\gamma/2,\rho/4}$. One can prove this fact by using trivial geometric arguments. From that property and taking into account that $S^- \in \mathcal{X}^-_{2,\gamma/4,\rho_0}$, it is straightforward to check that $\mathcal{U}^-$ so defined satisfies the property in (3.25). Now we look for solutions of (3.25) of the form $u(\tau) = \tau + u_1^-(\tau)$ and we obtain that u_1^- has to satisfy

$$\frac{d}{d\tau} u_1^- = \mathcal{U}^-(\tau + u_1^-) \tag{3.26}$$

We define the integral operator

$$\mathcal{I}^-(f)(\tau) = \int_{-\infty}^{\tau} f(t)\, dt$$

and we claim that for any $m > 1$ and $\gamma, \rho > 0$, $\mathcal{I}^- : \mathcal{X}_{m,\gamma,\rho} \to \mathcal{X}_{m-1,\gamma,\rho}$ and moreover $\|\mathcal{I}^-(f)\|_{m-1} \leq K_{\nu,\gamma}\|f\|_m$. Indeed, the proof is straightforward from (2.13) and Cauchy's theorem:

$$|\mathcal{I}^-(f)(\tau)| = \left| \int_{-\infty}^{0} f(\tau+t)\, dt \right| \leq \|f\|_m \int_{-\infty}^{0} \frac{1}{|\tau+t|^m} \leq \|f\|_m K_{m,\gamma} \frac{1}{|\tau|^{m-1}}.$$

We define $\mathcal{N}^-(f)(\tau) = \mathcal{U}^-(\tau + f(\tau))$ and we emphasize that, if u_1^- is a solution of the fixed point equation $u_1^- = \mathcal{I}^- \circ \mathcal{N}^-(u_1^-)$ then u_1^- satisfies equation (3.26). Since $\mathcal{U}^-$ belongs to $\mathcal{X}_{2,\gamma/2,\rho/4}$, we also have that $\mathcal{N}^-(0) \in \mathcal{X}^-_{2,\gamma/2,\rho/4}$. We can also check that, taking ρ big enough, $\tau + f(\tau) \in D^-_{\gamma/2,\rho/4}$ if $\tau \in D^-_{\gamma,\rho}$ and $f \in \mathcal{X}^-_{1,\gamma,\rho}$. Moreover, applying the mean value theorem and Lemma 2.6, then $\mathcal{N}^-(f) - \mathcal{N}^-(0) \in \mathcal{X}^-_{4,\gamma,\rho}$, taking ρ big enough. Hence we can conclude that the fixed point equation $u_1^- = \mathcal{I}^- \circ \mathcal{N}^-(u_1^-)$ has a solution belonging to $\mathcal{X}^-_{1,\gamma,\rho}$ following similar arguments as the ones given in the proof of Lemma 2.7. Finally we undo the change of variables and we obtain the result.

Now we deal with the statement of Δs. As the functions $S^\pm$ given in equation (3.23) are

$$S^\pm(s) = \bar{H}(\phi^\pm, \varphi^\pm, -s^{-1}),$$

it is clear that there exists $\Theta_0(s) \in \mathcal{Y}_0 \times \mathcal{Y}_0$ such that

$$S^-(s) - S^+(s) = \langle \Theta_0(s), \Delta\Phi(s) \rangle.$$

Moreover

$$S^-(s^-(\tau)) - S^-(s^+(\tau)) = \int_0^1 DS^-(s^+(\tau) + \lambda(s^-(\tau) - s^+(\tau))) \cdot \Delta s(\tau)\, d\lambda.$$

We note that, by Lemma 2.6, $DS^-(s) \in \mathcal{X}_{3,2\gamma,2\rho}$ provided $S^- \in \mathcal{X}_{2,\gamma,\rho}$, therefore,

$$S^-(s^-(\tau)) - S^-(s^+(\tau)) = \Theta_3(\tau)\Delta s(\tau)$$

with $\Theta_3 \in \mathcal{Y}_3$.

Now we subtract the equations that $s^\pm$ satisfy and we obtain that Δs satisfies the differential equation on $E_{\gamma,\rho}$

$$\begin{aligned} \Delta s'(\tau) &= S^-(s^-(\tau)) - S^+(s^+(\tau)) \\ &= S^-(s^-(\tau)) - S^-(s^+(\tau)) + S^-(s^+(\tau)) - S^+(s^+(\tau)) \\ &= \Theta_3(\tau)\Delta s(\tau) + \langle \Theta_0(s^+(\tau)), \Delta\Phi(s^+(\tau)) \rangle \end{aligned}$$

and therefore, since $\Delta s(\tau)\to 0$ as $\operatorname{Im}\tau\to-\infty$, we have that

$$\Delta s(\tau)=$$
$$\exp\Big(\int_{-\mathrm{i}\infty}^{\tau}\Theta_3(u)\,du\Big)\int_{-\mathrm{i}\infty}^{\tau}\exp\Big(-\int_{-\mathrm{i}\infty}^{t}\Theta_3(u)\,du\Big)\langle\Theta_0(s^+(t)),\Delta\Phi(s^+(t))\rangle\,dt.$$

Finally we observe that, as $\Theta_3\in\mathcal{Y}_3$, using Cauchy's theorem,

$$\begin{aligned}|\Delta s(\tau)|\le& C\int_{-\infty}^{0}|\langle\Theta_0(s^+(\tau+\mathrm{i}\,t)),\Delta\Phi(s^+(\tau+\mathrm{i}\,t))\rangle|\,dt\\ \le& C\|\Theta_0\|_{0,\times}\int_{-\infty}^{0}\|\Delta\Phi(s^+(\tau+\mathrm{i}\,t))\|\,dt.\end{aligned}\tag{3.27}$$

By Theorem 1.4 and taking into account expression (3.24) of s^+ and that $\beta(s)=-(c+\varepsilon\alpha h_0)\log(s(1+\varepsilon h_0 s^{-1}))$, there exists a constant K such that

$$\begin{aligned}\|\Delta\Phi(s^+(\tau+\mathrm{i}\,t))\|&\le K|s^+(\tau+\mathrm{i}\,t)\,\mathrm{e}^{-\mathrm{i}(\alpha s^+(\tau+\mathrm{i}\,t)+\beta(s^+(\tau+\mathrm{i}\,t)))}|\\ &\le K|\tau+\mathrm{i}\,t|\,\mathrm{e}^{\alpha t}\,|\,\mathrm{e}^{-\mathrm{i}(\alpha\tau-c\log(\tau+\mathrm{i}\,t)}|.\end{aligned}$$

Since $|\,\mathrm{e}^{\mathrm{i}\,c(\log\tau-\log(\tau+\mathrm{i}\,t))}|$ is bounded we have that, changing slightly K,

$$\|\Delta\Phi(s^+(\tau+\mathrm{i}\,t))\|\le K|\tau+\mathrm{i}\,t|\,\mathrm{e}^{\alpha t}\,|\,\mathrm{e}^{-\mathrm{i}(\alpha\tau-c\log\tau)}|.$$

Using this fact to bound (3.27) we obtain

$$|\,\mathrm{e}^{\mathrm{i}(\alpha\tau-c\log\tau)}\Delta s(\tau)|\le CK\|\Theta_0\|_{0,\times}\int_{-\infty}^{0}|\tau+\mathrm{i}\,t|\,\mathrm{e}^{\alpha t}\,dt\le CK\alpha^{-2}\|\Theta_0\|_{0,\times}|\tau|$$

and the second part of the lemma is proved. □

Proof of Theorem 1.2. Let $\Phi^\pm(s)=(\phi^\pm(s),\varphi^\pm(s))$ be the solutions of Theorem 1.4. We observe that $(\Phi^\pm(s^\pm(\tau)),s^\pm(\tau))$ are solutions of system (1.8). Henceforth $\Psi^\pm(\tau)=(\Phi^\pm(s^\pm(\tau)),-(s^\pm(\tau))^{-1})$ are solutions of system (1.1).

It is clear that, since $\Phi^\pm\in\mathcal{X}^\pm_{3,\gamma,\rho}\times\mathcal{X}^\pm_{3,\gamma,\rho}$, then, taking ρ big enough if necessary, $\pi^1\Psi^\pm,\pi^2\Psi^\pm\in\mathcal{X}^\pm_{3,2\gamma,2\rho}$. Moreover

$$\pi^3\Psi^\pm(\tau)=-\frac{1}{\tau+\varepsilon h_0\log\tau+(\varepsilon h_0)^2\tau^{-1}\log\tau+\mathcal{S}^\pm(\tau)}=-\frac{1}{\tau}+O(\tau^{-2}\log\tau).$$

On the other hand,

$$\begin{aligned}\pi^{1,2}(\Psi^-(\tau)-\Psi^+(\tau))&=\Phi^-(s^-(\tau))-\Phi^+(s^+(\tau))\\ &=\Phi^-(s^-(\tau))-\Phi^+(s^-(\tau))+\Phi^+(s^-(\tau))-\Phi^+(s^+(\tau))\\ &=\Delta\Phi(s^-(\tau))+\int_0^1\partial_s\Phi^+(s^+(\tau)+\lambda\Delta s(\tau))\Delta s(\tau)\,d\lambda.\end{aligned}$$

We note that by Lemma 3.6, $\partial_s\Phi^+\in\mathcal{Y}_{4,2\gamma,2\rho}\times\mathcal{Y}_{4,2\gamma,2\rho}$. Hence, by Lemma 3.7

$$\int_0^1\partial_s\Phi^+(s^+(\tau)+\lambda\Delta s(\tau))\Delta s(\tau)\,d\lambda=\bar\xi(\tau)\,\mathrm{e}^{-\mathrm{i}(\alpha\tau-c\log\tau)}$$

with $\bar\xi\in\mathcal{Y}_{3,2\rho,2\gamma}\times\mathcal{Y}_{3,2\rho,2\gamma}$. By Theorem 1.4 and Lemma 3.7, we obtain that

$$\begin{aligned}\pi^{1,2}(\Psi^-(\tau)-\Psi^+(\tau))=&s^-(\tau)\,\mathrm{e}^{-\mathrm{i}(\alpha s^-(\tau)+\beta(s^-(\tau)))}\varepsilon(C(\varepsilon)+\xi(s^-(\tau)))\\ &+\bar\xi(\tau)\,\mathrm{e}^{-\mathrm{i}(\alpha\tau-c\log\tau)}\\ =&\tau\,\mathrm{e}^{-\mathrm{i}(\alpha\tau-c\log\tau)}(\varepsilon C(\varepsilon)+\tilde\xi(\tau))\end{aligned}$$

where $\tilde{\xi} = O(\tau^{-1}\log\tau)$ and $C(\varepsilon)$ is the function defined in Theorem 1.4.

We know that if the constant $C(\varepsilon)$ is zero one has that $\Delta\Phi = 0$. But in this case it is easy to see that we also have $\Delta s = 0$, and therefore $\Delta\Psi = 0$.

We also observe that

$$\pi^3(\Psi^-(\tau) - \Psi^+(\tau)) = \frac{1}{s^-(\tau)} - \frac{1}{s^+(\tau)} = -\frac{\Delta s(\tau)}{s^-(\tau)\cdot s^+(\tau)} = \frac{1}{\tau}\,\mathrm{e}^{-\,\mathrm{i}(\alpha\tau - c\log\tau)}\,h(s)$$

where h is a bounded function in $E_{2\gamma,2\rho}$.

Finally we point out that, if $\varepsilon = 0$, again we have that $\Delta\Phi = 0$ which implies that $\Delta\Psi = 0$ and the proof is complete. □

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