Numerical computation of the asymptotic size of the rotation domain for the Arnold family

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Abstract

We consider the Arnold Tongue of the Arnold family of circle maps associated to a fixed Diophantine rotation number θ. The corresponding maps of the family are analytically conjugate to a rigid rotation. This conjugation is defined on a (maximal) complex strip of the circle and, after a suitable scaling, the size of this strip is given by an analytic function of the perturbative parameter.

The main purpose of this paper is to perform a numerical accurate computation of this function and of its Taylor expansion. This allows us to verify previous theoretical results. The rotation numbers we select are quadratic irrationals, mainly the Golden Mean.

By introducing a nonstandard extrapolation process, specially suited for the problem, we compute all the quantities required (rotation numbers, Arnold Tongues, Fourier and Taylor coefficients) with high precision.

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Contents

1 Introduction

In this paper we consider the widely studied *Arnold family* of circle maps,

$$\begin{array}{rccc} \tilde{f}_{\alpha,\varepsilon}: & \mathbb{T}^1 & \longrightarrow & \mathbb{T}^1 \\ & x & \longrightarrow & x + \frac{\alpha}{2\pi} + \frac{\varepsilon}{2\pi}\sin(2\pi x) \end{array} \tag{1}$$

being $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$, where (α, ε) are real parameters. For any $\alpha \in [0, 2\pi)$ and $\varepsilon \in [0, 1)$, the map $\tilde{f}_{\alpha,\varepsilon}$ is an orientation-preserving analytic diffeomorphism of the circle and we denote by $\rho(\alpha, \varepsilon)$ its rotation number.

A well-known result on circle maps [1, 7, 9] ensures that, given f an analytic diffeomorphism of $\mathbb{T}^1$, whose rotation number $\theta = \rho(f)$ is Diophantine, then the map f is analytically conjugate to the rigid rotation $\mathcal{T}_\theta(x) = x + \theta$. Concretely, there exists an analytic diffeomorphism $\eta : \mathbb{T}^1 \to \mathbb{T}^1$ such that $\eta \circ \mathcal{T}_\theta = f \circ \eta$. If we require $\eta(0) = x_0$, for a fixed $x_0 \in \mathbb{T}^1$, then the conjugacy is unique. This conjugation can be written as

$$\eta(x) = x + \xi(x), \tag{2}$$

where ξ is a 1-periodic function. As η is (real) analytic, it can be analytically extended to a maximal complex strip of the form

$$\mathcal{A}(\Delta) = \{x \in \mathbb{C}/\mathbb{Z} : |\mathrm{Im}(x)| < \Delta\}, \tag{3}$$

for certain $\Delta > 0$. Abusing notation, we also denote by η this analytic extension. By the principle of analytic continuation, the map η still conjugates f to $\mathcal{T}_\theta$ in $\mathcal{A}(\Delta)$.

To apply this result to the Arnold family, we have to take into account the parametric dependence. Thus, given an arbitrary $\theta \in [0, 1)$, the set $T_\theta = \{(\alpha, \varepsilon) : \rho(\alpha, \varepsilon) = \theta\}$ is called the *Arnold Tongue* of $\tilde{f}_{\alpha,\varepsilon}$ of rotation number θ. If θ is a Diophantine number, then T_θ is an analytic curve which is the graph of a function $\varepsilon \in [0, 1) \mapsto \alpha(\varepsilon)$, with $\alpha(0) = 2\pi\theta$ (see [14]). Hence, if we keep the Diophantine number θ fixed from now on, we have that the 1-parameter family of maps $\tilde{f}_{\alpha(\varepsilon),\varepsilon}$ is analytically conjugate to $\mathcal{T}_\theta$ through a family of analytic conjugations,

$$\tilde{\eta}_\varepsilon : \mathcal{A}(\tilde{\Delta}(\varepsilon)) \to \mathbb{C}, \tag{4}$$

depending also analytically on ε. Here, $\mathcal{A}(\tilde{\Delta}(\varepsilon))$ denotes the maximal strip in which $\tilde{\eta}_\varepsilon$ is defined.

For $\tilde{\Delta}(\varepsilon)$ we easily have that $\lim_{\varepsilon\to 1^-} \tilde{\Delta}(\varepsilon) = 0$ and $\lim_{\varepsilon\to 0^+} \tilde{\Delta}(\varepsilon) = +\infty$. In this paper we focus on the asymptotic behavior of $\tilde{\Delta}(\varepsilon)$ when $\varepsilon \to 0^+$. This problem was first considered in [3], where an asymptotic expression for this function was given. Concretely, if we write

$$\tilde{\Delta}(\varepsilon) = \frac{1}{2\pi}\log \tilde{R}_\varepsilon, \tag{5}$$

it was proved that

$$\tilde{R}_\varepsilon = \frac{2}{\varepsilon}R_\varepsilon = \frac{2}{\varepsilon}(R_0 + O(\varepsilon\log\varepsilon)). \tag{6}$$

Here, R_0 is the conformal radius of the *Siegel disk* at the origin of the so-called *complex semistandard map*

$$G(z) = z\mathrm{e}^{\mathrm{i}\omega}\mathrm{e}^z, \tag{7}$$

where $\omega = 2\pi\theta$. Indeed, there exists a unique analytic diffeomorphism

$$\varphi : \mathbb{D}_{R_0} \to \mathbb{C} \tag{8}$$

such that $\varphi(0) = 0$, $\varphi'(0) = 1$ and $\varphi \circ \mathcal{R}_\omega = G \circ \varphi$, where $\mathbb{D}_{R_0} = \{z \in \mathbb{C} : |z| < R_0\}$ and $\mathcal{R}_\omega(z) = \mathrm{e}^{\mathrm{i}\omega} z$.

The estimate (6) was later improved in [2], where the authors proved that R_ε is an even analytic function in the unit disk $\mathbb{D}_1$, so that

$$\tilde{R}_\varepsilon = \frac{2}{\varepsilon}(R_0 + O(\varepsilon^2)). \tag{9}$$

It is not difficult to give a geometrical view of this result. Let us consider an analytic map $F : \mathcal{U} \subset \mathbb{C} \to \mathbb{C}$ leaving the unit circle $\mathbf{C}_1$ invariant. We say that $F_{|\mathbf{C}_1}$ is *analytically linearizable* if there exists an analytic diffeomorphism $\varphi : \mathbf{C}_1 \to \mathbf{C}_1$, such that $\varphi \circ \mathcal{R}_\omega = F \circ \varphi$. If we ask $\varphi(1) = z_0$, for some $z_0 \in \mathbf{C}_1$, then φ is univocally defined. Being φ an analytic function of $\mathbf{C}_1$, it can be analytically continued to a maximal annulus around $\mathbf{C}_1$ of the form

$$A(1/R, R) = \{z \in \mathbb{C} : 1/R < |z| < R\}, \tag{10}$$

for some $R > 1$. Now, we consider $f : \mathbb{T}^1 \to \mathbb{T}^1$ the (analytic) circle map induced by $F_{|\mathbf{C}_1}$, using the exponential map $z = \mathrm{e}^{2\pi\mathrm{i}x}$, and we define $\eta : \mathbb{T}^1 \to \mathbb{T}^1$ so that

$$\varphi(\mathrm{e}^{2\pi\mathrm{i}x}) = \mathrm{e}^{2\pi\mathrm{i}\eta(x)}, \qquad x \in \mathbb{T}^1, \tag{11}$$

with the normalization $\eta(0) = x_0$, where $\mathrm{e}^{2\pi\mathrm{i}x_0} = z_0$. Then, we have that $\eta \circ \mathcal{T}_\theta = f \circ \eta$, and thus, f is analytically conjugate to a rotation. Moreover, η is also analytic and the width of its strip of analyticity around $\mathbb{T}^1$ is $\Delta = (1/2\pi)\log R$. The image by φ of the maximal annulus $A(1/R, R)$ where φ can be analytically continued is called the *Herman ring* of F.

Remark 1. *We use the term* rotation domain *to refer to the image of the maximal domain of definition of an analytic conjugation to a rigid rotation of a circle map. We extend the term to refer to a Siegel disk or a Herman ring of an analytic map of* $\mathbb{C}$*, when there is no danger of confusion. For a Herman ring, we call* size of the ring *to the outer radius* R *of the annulus (10) where the conjugation is defined. Similarly, for a Siegel disk we use its conformal radius* R_0 *of (8) to size up the disk and, for a circle map, we measure the size of its rotation domain in terms of the width* Δ *of the strip of analyticity of the conjugation (3).*

Now, we consider the *complex standard family*

$$\tilde{F}_{\alpha,\varepsilon}(z) = z\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{\frac{\varepsilon}{2}(z-\frac{1}{z})}, \qquad \alpha \in [0, 2\pi), \quad \varepsilon \in [0, 1). \tag{12}$$

This is a family of holomorphic maps of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ leaving $\mathbf{C}_1$ invariant. In $\mathbb{T}^1$, this family of maps induces the Arnold family (1). Thus, the geometrical meaning of formula (9) is that, by means of a suitable scaling, the complex standard family becomes the semistandard map (7) when $(\alpha, \varepsilon) \to (\omega, 0)$ over T_θ, and the Herman ring of $\tilde{F}_{\alpha(\varepsilon),\varepsilon}$ becomes the Siegel disk of G (see [3]).

The main purpose of this paper is to perform a numerical verification of the asymptotic formula (9), working with the standard family $\tilde{f}_{\alpha,\varepsilon}$. The rotation numbers we select for the computations are quadratic irrationals, mainly focusing in the case when θ is the Golden Mean, $\theta = (\sqrt{5} - 1)/2$.

To compute the width of the strip of analyticity of the conjugation we are going to use a result due to Herman (see Proposition 3). Moreover, as

$$\Delta(\varepsilon) = \frac{1}{2\pi}\log R_\varepsilon = \tilde{\Delta}(\varepsilon) - \frac{1}{2\pi}\log\left(\frac{2}{\varepsilon}\right) \tag{13}$$

is an analytic (even) function of ε, we also adapt Herman's method to compute the Taylor expansion of $\Delta(\varepsilon)$ at $\varepsilon = 0$. Next to that, we compare this Taylor expansion with $\Delta(\varepsilon)$ computing this function for a table of values of $\varepsilon \in [0,1]$.

Among the problems we have faced to perform these numerical computations, with enough precision to make a successful comparison with the Taylor expansion, here we want to stress two. First, the accurate computation of the Arnold Tongue T_θ. For this purpose, we have used a numerical method previously developed by the authors to compute rotation numbers with high precision (see [15], Section 4.2). Second, the improvement of the numerical results for $\Delta(\varepsilon)$ provided by Herman's method. To do that, we have combined the direct computation with some heuristic observations and semi-analytical ideas, in order to develop an extrapolation process suited *ad hoc* for the method, that depends strongly on the arithmetic properties on the selected rotation numbers.

To give a partial justification of these ideas, for the case of the Golden Mean, we have also adapted Herman's method for computing the Fourier coefficients of the periodic part of the conjugation (2).

We also mention that all the numerical computations have been implemented *ad hoc* in C++ code. Moreover, in order to perform the computations of the different quantities with enough precision to detect its asymptotic behavior, we have replaced the standard *double* data type of the computer by the so-called *double-double* data type, of approximately 32 decimal digits, which is provided by the *quad-double/double-double computational package* (see [11]).

The contents of the paper are as follows. In Section 2 we present Herman's result and show how it can be used to compute the function $\Delta(\varepsilon)$ as well as its derivatives. Moreover, in Section 2.3 we adapt the previous method for computing the Fourier coefficients of the conjugation. Section 3 is devoted to apply this methodology to the Arnold family. For the case of the Golden Mean, in Section 3.3 we develop an extrapolation method to improve Herman's method. Moreover, we also give numerical evidences of the correctness of the asymptotic expansions used in this extrapolation process. In Section 3.4 we compute some Fourier coefficients of the conjugation and detect its asymptotic behavior. This behavior is used in Appendix B to give a partial justification of the asymptotics used in Section 3.3. In Section 3.5 we briefly discuss the case of other quadratic irrational rotation numbers. Finally, in Appendix A, we analytically compute some Taylor coefficients of the function $\alpha(\varepsilon)$ for any Diophantine rotation number θ. These coefficients are required in Section 3.2.

2 Computation of the size of the rotation domain

In Section 2.1 we introduce Herman's method to compute the size of a Siegel disk or a Herman ring of a map in the complex plane. Our next step is to translate this method in order to compute the size Δ of the rotation domain of a circle map. Later, in Section 2.2, we formulate this method in terms of a one-parameter family of circle maps f_μ, so that we can adapt it to the computation of the derivatives of the size $\Delta(\mu)$. Finally, in Section 2.3, a slightly modification of the method is used to compute the Fourier coefficients of the conjugation.

2.1 Herman's method

Let F be an analytic map of $\mathbb{C}$, leaving $\mathbf{C}_1$ invariant, and f the induced map on $\mathbb{T}^1$ via the complex exponential, which we suppose is a circle analytic diffeomorphism.

Remark 2. *In the forthcoming we are going to deal with a lift of $F_{|\mathbf{C}_1}$ to $\mathbb{R}$ rather than the corresponding map on $\mathbb{T}^1$. Thus, from now on we denote by f this lift, in the understanding that, to define the corresponding map on $\mathbb{T}^1$, we only have to take modulo one in the formula of f. This construction is straightforward for the Arnold family.*

We suppose that $F_{|\mathbf{C}_1}$ has a Diophantine rotation number θ, and we want to discuss how to compute the size R of its Herman ring and the size Δ of the rotation domain of f (see Remark 1). If we focus for instance on the definition of R, what we have to do, in principle, is to compute the Laurent expansion of the conjugation φ around $\mathbf{C}_1$ (see (11)). Then, we can obtain its outer radius of convergence from the behavior of the coefficients of this expansion. Of course, it is not realistic to expect that, by applying this method to $\tilde{F}_{\alpha(\varepsilon),\varepsilon}$ in (12), we can obtain a numerical approximation to $\tilde{R}_\varepsilon$ with enough precision to detect its asymptotic behavior (9).

Alternatively, we proceed analogously as Marmi in [13], and use the following result due to Herman.

Proposition 3 (Herman, [10]). *Let F be an analytic map in a neighbourhood of the origin, such that $F(0) = 0$ and $F'(0) = \mathrm{e}^{2\pi\mathrm{i}\theta}$. If F is linearizable with linearization φ (see (8)), and we pick up $z = \varphi(w)$, with $|w| = r < R$, where R is the conformal radius of its Siegel disk U, we have that $z \in U$ and that*

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log |F^j(z)| = \log r. \tag{14}$$

Moreover, if $\{p_j/q_j\}_{j\geq 0}$ are the convergents of the continuous fraction expansion of θ, then

$$\left| \frac{1}{q_j} \sum_{j=0}^{q_j-1} \log |F^j(z)| - \log r \right| \leq \frac{1}{q_j} \mathrm{var}(\log |\varphi_{|\mathbf{C}_r}|), \tag{15}$$

where $\mathrm{var}(\cdot)$ *is the variation of the curve.*

This result can be generalized to the case of Herman rings of complex maps (see [13]). Moreover, if we suppose that we are able to take the limit when $r \to R^-$, then we can use (14) and (15) to compute R by taking $z \in \partial U$.

Let us display what Proposition 3 means in terms of the circle map f and the size of its rotation domain Δ. We consider a point $a - \mathrm{i}\Delta$ on the (lower) boundary of the strip of analyticity of the conjugation η in (2), with $a \in \mathbb{R}$, and we iterate $x^* = \eta(a - \mathrm{i}\Delta)$ (assuming this point defined) by the action of f. By expanding ξ in Fourier series,

$$\xi(x) = \sum_{k\in\mathbb{Z}} \xi_k \mathrm{e}^{2\pi\mathrm{i}kx}, \tag{16}$$

we obtain

$$f^n(x^*) = \eta(a - \mathrm{i}\Delta + n\theta) = a - \mathrm{i}\Delta + n\theta + \sum_{k\in\mathbb{Z}} \xi_k \mathrm{e}^{2\pi\mathrm{i}k(a-\mathrm{i}\Delta+n\theta)}.$$

Let us note that, being ξ a real analytic function, its Fourier coefficients verify $\xi_{-k} = \bar{\xi}_k$, for any $k \in \mathbb{Z}$. In particular, $\xi_0 \in \mathbb{R}$.

Now, if we denote by $\hat{\xi}_k = \xi_k \mathrm{e}^{2\pi\mathrm{i}k(a-\mathrm{i}\Delta)}$, for $k \neq 0$, $\hat{\xi}_0 = \xi_0 + a - \mathrm{i}\Delta$ and $\hat{f}_n = f^n(x^*) - n\theta$, we have

$$\hat{f}_n = \sum_{k\in\mathbb{Z}} \hat{\xi}_k \mathrm{e}^{2\pi\mathrm{i}kn\theta}. \tag{17}$$

In view of Proposition 3, we consider the sum of the first N iterates of the map

$$S_N = \sum_{n=0}^{N-1} \hat{f}_n = N\hat{\xi}_0 + \sum_{k\in\mathbb{Z}\setminus\{0\}} \hat{\xi}_k \frac{1-\mathrm{e}^{2\pi \mathrm{i} kN\theta}}{1-\mathrm{e}^{2\pi \mathrm{i} k\theta}}. \tag{18}$$

Hence, by assuming that the sum at the right-hand side of (18) divided by N goes to zero when $N \to +\infty$, we recover Herman's result

$$\lim_{N\to+\infty} \mathrm{Im}\left(\frac{S_N}{N}\right) = -\Delta. \tag{19}$$

Remark 4. *We point out that the most faster convergence speed we can expect for Δ in (19) is of $O(1/N)$, i. e., the same order of convergence expected when computing the rotation number of a circle map from its definition. Later, in Section 3.3, we are going to discuss how this convergence can be accelerated (for the Arnold family and θ being the Golden Mean) by means of a suitable extrapolation process.*

The main difficulty of using (19) for computing Δ lies on knowing a point x^* on the boundary of the rotation domain of the circle map f. The most natural candidates are the critical points of the map, defined so that $f'(x^*) = 0$. It is clear that a critical point cannot be in the interior of any rotation domain, and a very important problem is to investigate if there is a critical point on its boundary. Herman showed in [10] that there are examples of maps without critical points on the boundary of their Siegel disk. However, there are several results in the positive direction (see for instance [5, 8, 6]). For our concerns, Geyer claimed in [4] that the critical points of $\tilde{F}_{\alpha,\varepsilon}$ in (12) are always on the boundary of its Herman ring for rotation numbers θ of constant type[1] (the same also holds for the Siegel disk of the semistandard map G in (7)).

2.2 Variationals of Herman's method

Now we consider a parametric approach to formula (19). Let us suppose that $f_\mu : \mathbb{R} \to \mathbb{R}$ is a one-parameter family of real analytic maps, which are lifts of a one-parameter family of diffeomorphisms of the circle. We also suppose that the rotation number $\theta = \rho(f_\mu)$ is independent of μ and Diophantine. If the dependence on μ of the family f_μ is smooth enough (analytic in our context), one can ask if the function $\Delta(\mu)$ giving the size of the rotation domain of f_μ is also smooth. Assuming the answer positive, one can try to use formula (19) to compute the derivatives of $\Delta(\mu)$. For this purpose, we suppose known, for any μ, a (complex) point x_μ^* at the (lower) boundary of the rotation domain of f_μ. We also suppose that x_μ^* depends smoothly on μ (from the practical point of view x_μ^* has to be a critical point of the map f_μ). Hence, we have

$$\lim_{N\to+\infty} \mathrm{Im}\left(\frac{1}{N}\sum_{n=0}^{N-1}(f_\mu^n(x_\mu^*) - n\theta)\right) = -\Delta(\mu).$$

Then, by taking derivatives with respect to μ, we obtain the following (formal) expressions

$$\lim_{N\to+\infty} \mathrm{Im}\left(\frac{1}{N}\sum_{n=0}^{N-1}\frac{d^k}{d\mu^k}(f_\mu^n(x_\mu^*) - n\theta)\right) = -\Delta^{(k)}(\mu), \qquad k \geq 0, \tag{20}$$

where the derivatives of $f_\mu^n(x_\mu^*)$ can be computed recurrently (see Section 3.2).

[1] $\theta \in \mathbb{R}\setminus\mathbb{Q}$ is of constant type if its continuous fraction expansion, $\theta = [a_0; a_1, a_2, \ldots]$, verifies $a_0 \in \mathbb{Z}$, $a_n \in \mathbb{N}$ and $\sup_n a_n < +\infty$.

2.3 Fourier coefficients of the conjugation

Sometimes it could be useful to compute the Fourier coefficients of $\xi(x)$ in (16). See for instance Section 3.4 for the case of the Arnold family.

For this purpose, we focus on formula (17) and on the modified Fourier coefficients $\hat{\xi}_k$, which are those of the Fourier expansion of ξ at the lower boundary of its domain of analyticity. The most natural method to compute them numerically is to truncate (17), and to consider the approximate linear relation thus obtained

$$\hat{f}_n \approx \sum_{|k|\le K} \hat{\xi}_k \mathrm{e}^{2\pi \mathrm{i} k n\theta},$$

for certain $K > 0$. Thus, by computing a finite number of iterates of the map, we can obtain numerical approximations for $\hat{\xi}_k$, $|k| \le K$, by solving a linear system. We observe that the matrix of this system is Vandermonde-like, whose determinant is given by the product of quantities of the form $\mathrm{e}^{2\pi \mathrm{i} k\theta} - \mathrm{e}^{2\pi \mathrm{i} k'\theta} = \mathrm{e}^{2\pi \mathrm{i} k\theta}(1 - \mathrm{e}^{2\pi \mathrm{i}(k'-k)\theta})$. This means that the determinant is obtained from a product of "small divisors", which can lead to an ill-conditioned system of equations.

In this section we discuss an alternative method, based on a modification of the definition of S_N in (18), allowing to compute those Fourier coefficients in the same way as Δ from (19). Given a fixed $k^* \in \mathbb{Z}$, we denote by $\hat{f}_n^{k^*} = \hat{f}_n \mathrm{e}^{-2\pi \mathrm{i} k^* n\theta}$. Hence, from (17) we have,

$$\hat{f}_n^{k^*} = \sum_{k\in\mathbb{Z}} \hat{\xi}_k \mathrm{e}^{2\pi \mathrm{i}(k-k^*)n\theta}.$$

In this case, the sum of these modified iterates gives

$$S_N^{k^*} = \sum_{n=0}^{N-1} \hat{f}_n^{k^*} = N\hat{\xi}_{k^*} + \sum_{k\in\mathbb{Z}\backslash\{k^*\}} \hat{\xi}_k \frac{1 - \mathrm{e}^{2\pi \mathrm{i}(k-k^*)N\theta}}{1 - \mathrm{e}^{2\pi \mathrm{i}(k-k^*)\theta}}. \tag{21}$$

Then, under the same assumptions on the limit we made in (19), we obtain

$$\lim_{N\to+\infty} \frac{S_N^{k^*}}{N} = \hat{\xi}_{k^*}. \tag{22}$$

Remark 5. *If f_μ is the one-parameter family of maps of Section 2.2, then we can compute the derivatives of $\hat{\xi}_{k^*}(\mu)$ by differentiating formula* (22) *analogously as we did in (20).*

Remark 6. *The direct evaluation of* $\mathrm{e}^{2\pi \mathrm{i} k\theta} = \cos(2\pi k\theta) + \mathrm{i}\sin(2\pi k\theta)$, *for $k \ge 1$, is very "expensive" from the numerical point of view, but we observe that* $\cos(2\pi k\theta)$ *and* $\sin(2\pi k\theta)$ *can be computed recursively by using a recurrence that is numerically stable. Thus, we only need to compute* $\cos(2\pi\theta)$ *and* $\sin(2\pi\theta)$.

Formula (22) provides the coefficient $\hat{\xi}_{k^*}$ for any $k^* \in \mathbb{N}$ (coefficients with $k < 0$ can be easily obtained from those with $k > 0$ and are exponentially small in $|k|$, of $O(\mathrm{e}^{-4\pi\Delta|k|})$). However, we observe that (22) converges slowly as we increase k^*. The reason is that, for k^* "big", there are many coefficients $\hat{\xi}_k$ with $0 \le k << k^*$ which have bigger size than $\hat{\xi}_{k^*}$. To overcome this problem one can use the following trick. First, we use (22) to compute $\hat{\xi}_k$ for $0 \le k \le K$, with K not "too big". From the numerical approximations thus obtained, namely $\{\bar{\xi}_k\}_{0\le k\le K}$, we consider again formula (22), but now applied to

$$\bar{f}_n^{k^*} = \hat{f}_n^{k^*} - \sum_{0\le k\le K} \bar{\xi}_k \mathrm{e}^{2\pi \mathrm{i}(k-k^*)n\theta}.$$

This new expression can be used even to improve the numerical approximations $\{\bar{\xi}_k\}_{0\leq k\leq K}$ or to compute new coefficients with $k > K$. Of course, this process can be iterated but, unfortunately, this is more expensive than the direct method (22).

Nevertheless, in this paper we use another approach in order to improve $\hat{\xi}_k$. In Section 3.4, we apply formula (22) to compute these Fourier coefficients for the Arnold family (1), when the rotation number is the Golden Mean. Then, the experimental study of these coefficients gives us the chance to apply an extrapolation process to refine them.

3 Application to the Arnold family

In this section we consider the methods of Section 2 for the case when the map F is $\tilde{F}_{\alpha(\varepsilon),\varepsilon}$ in (12), and thus $f = \tilde{f}_{\alpha(\varepsilon),\varepsilon}$ in (1), where $\alpha = \alpha(\varepsilon)$ is the parameterization of an Arnold Tongue T_θ for the Arnold family, for a fixed Diophantine number θ.

In the numerical experiments we display along this section we take θ the Golden Mean. Finally, in Section 3.5, we explore the case of other quadratic irrational rotation numbers.

The map $\tilde{F}_{\alpha,\varepsilon}$ has two critical points located at

$$z^*_\pm = \frac{1}{\varepsilon}(-1 \pm \sqrt{1-\varepsilon^2}) < 0.$$

If we use the transformation $z = \mathrm{e}^{2\pi \mathrm{i} x}$, we obtain the critical points of $\tilde{f}_{\alpha,\varepsilon}$:

$$x^*_\pm(\varepsilon) = \frac{1}{2} - \frac{\mathrm{i}}{2\pi} \log\left(\frac{1 \pm \sqrt{1-\varepsilon^2}}{\varepsilon}\right).$$

As we are interested in the critical point on the lower boundary, we pick up $x^* = x^*_+(\varepsilon)$.

3.1 Scaling the Arnold family

The first problem we face when trying to compute the asymptotic size of the rotation domain of $\tilde{f}_{\alpha(\varepsilon),\varepsilon}$ is that the function $\tilde{\Delta}(\varepsilon)$ in (5) is not bounded when $\varepsilon \to 0$. Nevertheless, as we know *a priori* that $\Delta(\varepsilon)$ in (13) is analytic in $\mathbb{D}_1$, we perform a scaling on the Arnold family to focus on the computation of $\Delta(\varepsilon)$.

Thus, we introduce the change of variables $x = t - (\mathrm{i}/2\pi)\log(2/\varepsilon)$ and denote by $f_{\alpha,\varepsilon}$ the Arnold family $\tilde{f}_{\alpha,\varepsilon}$ expressed in this new variable,

$$f_{\alpha,\varepsilon}(t) = t + \frac{\alpha}{2\pi} - \frac{\mathrm{i}}{2\pi}\mathrm{e}^{2\pi \mathrm{i} t} + \varepsilon^2 \frac{\mathrm{i}}{8\pi}\mathrm{e}^{-2\pi \mathrm{i} t}. \tag{23}$$

The map $f_{\alpha(\varepsilon),\varepsilon}$ in analytically conjugate to the rotation $\mathcal{T}_\theta$ through the (scaled) conjugation

$$\eta_\varepsilon(t) = \tilde{\eta}_\varepsilon\left(t - \frac{\mathrm{i}}{2\pi}\log\left(\frac{2}{\varepsilon}\right)\right) + \frac{\mathrm{i}}{2\pi}\log\left(\frac{2}{\varepsilon}\right), \tag{24}$$

defined in the (maximal) complex strip (see (4))

$$\mathbf{A}(\varepsilon) = \left\{ t \in \mathbb{C}/\mathbb{Z} : \ -\Delta(\varepsilon) < \mathrm{Im}(t) < \Delta(\varepsilon) + \frac{1}{\pi}\log\left(\frac{2}{\varepsilon}\right)\right\}.$$

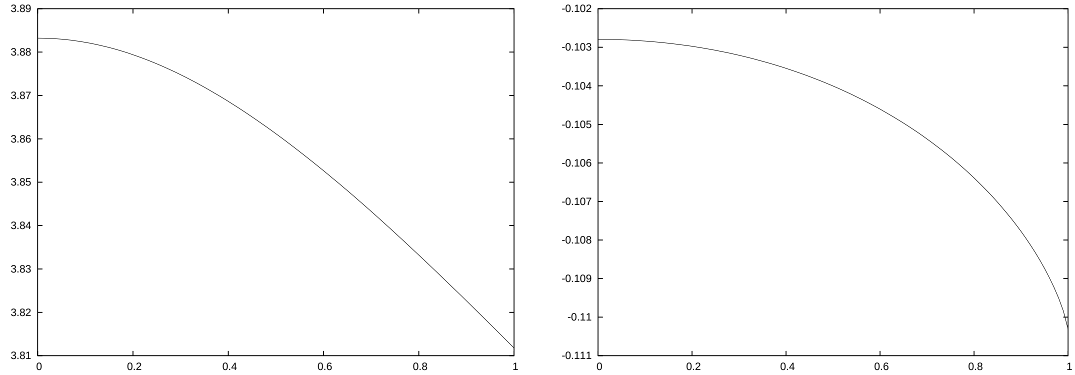

Figure 1: Left: the graph of $\alpha(\varepsilon)$ for the Arnold Tongue of the Arnold family for the Golden Mean. Right: the graph of $\Delta(\varepsilon)$ using (25).

So, we apply Herman's method to compute the lower border of this strip,

$$\lim_{N\to+\infty} \operatorname{Im}\left(\frac{1}{N}\sum_{n=0}^{N-1}(f^n_{\alpha(\varepsilon),\varepsilon}(t^*_\varepsilon)-n\theta)\right) = -\Delta(\varepsilon), \tag{25}$$

where

$$t^*_\varepsilon = \frac{1}{2} - \frac{\mathrm{i}}{2\pi}\log\left(\frac{1+\sqrt{1-\varepsilon^2}}{2}\right) \tag{26}$$

is now the lower critical point of $f_{\alpha(\varepsilon),\varepsilon}$.

In Figure 1 it is plotted the function $\Delta(\varepsilon)$ obtained for the case of the Golden mean using $N = F_{34}$ iterates of the map, where $F_{34} = 9227465$ is a Fibonacci number (see Section 3.3 for the motivation). We remark that, to perform these computations, we need a previous knowledge of the function $\alpha(\varepsilon)$, giving the Arnold Tongue, with enough precision. This precision is important to avoid big effects of the propagation of the error after doing a big number of iterates of the map. The function $\alpha(\varepsilon)$ has been obtained using a method introduced in [15] for computing the rotation number of a circle map with high precision (see Section 3.3 for a brief explanation of the method) and the secant method. See also [12] for a similar approach using the Newton method. Using [15], the function $\alpha(\varepsilon)$ has been computed (numerically) so that the rotation number of the points on "the tongue" is the Golden Mean with an (estimated) error smaller than 10^{-32}. The graph of $\alpha(\varepsilon)$ is also plotted in Figure 1.

3.2 Explicit recurrences for the scaled map

Our purpose now is to apply the method of Section 2 to compute the Taylor expansion of $\Delta(\varepsilon)$. As all the quantities we are going to consider turn to be even with respect to ε, we introduce a new parameter $\mu = \varepsilon^2$. Abusing notation, in the rest of this section we are going to write $\Delta(\mu)$ instead of $\Delta(\varepsilon)$ and the same for the other ε-depending quantities. In this way, we introduce

$$f_\mu(t) = f_{\alpha(\varepsilon),\varepsilon}(t) = t + \frac{\alpha(\mu)}{2\pi} - \frac{\mathrm{i}}{2\pi}\mathrm{e}^{2\pi \mathrm{i}t} + \mu\frac{\mathrm{i}}{8\pi}\mathrm{e}^{-2\pi \mathrm{i}t}, \tag{27}$$

whose lower critical point is

$$t^*_\mu = \frac{1}{2} - \frac{\mathrm{i}}{2\pi}\log\left(\frac{1+\sqrt{1-\mu}}{2}\right). \tag{28}$$

We focus on the first three coefficients of the Taylor expansion of $\Delta(\mu)$

$$\Delta(\mu) = \delta_0 + \mu\delta_1 + \mu^2\delta_2 + \cdots , \tag{29}$$

where $\delta_k = \Delta^{(k)}(0)/k!$.

The computation of δ_0 follows by applying (19) to the *semistandard map in the circle*, which is obtained through the identification $G(\mathrm{e}^{2\pi \mathrm{i}t}) = \mathrm{e}^{2\pi \mathrm{i}g(t)}$ (see (7)), and it is given by the expression

$$g(t) = f_0(t) = t + \theta - \frac{\mathrm{i}}{2\pi}\mathrm{e}^{2\pi \mathrm{i}t} \tag{30}$$

(recall $\alpha(0) = 2\pi\theta$), being its critical point $t_0^* = 1/2$. Then, we have

$$\delta_0 = \lim_{N\to+\infty} \mathrm{Im}\left(\frac{S_N}{N}\right) = \lim_{N\to+\infty} \mathrm{Im}\left(\frac{1}{N}\sum_{n=0}^{N-1}\hat{g}_n\right), \tag{31}$$

where $\hat{g}_n = g^n(1/2) - n\theta$.

To compute $\Delta^{(k)}(0)$ for $k = 1, 2$, we use (20). So, the main point is to obtain the derivatives of the iterates of the critical point, $\frac{d^k}{d\mu^k}\left(f_\mu^n(t_\mu^*)\right)_{|\mu=0}$, which can be computed recursively. More precisely, we introduce

$$u_n(\mu) = f_\mu^n(t_\mu^*), \quad v_n(\mu) = u_n'(\mu), \quad w_n(\mu) = u_n''(\mu),$$

and then from (27) we obtain the following recurrences:

$$\begin{aligned}
u_{n+1}(\mu) &= f_\mu(u_n(\mu)) = u_n(\mu) + \frac{\alpha(\mu)}{2\pi} - \frac{\mathrm{i}}{2\pi}\mathrm{e}^{2\pi \mathrm{i}u_n(\mu)} + \mu\frac{\mathrm{i}}{8\pi}\mathrm{e}^{-2\pi \mathrm{i}u_n(\mu)},\\
v_{n+1}(\mu) &= v_n(\mu) + \frac{\alpha'(\mu)}{2\pi} + \mathrm{e}^{2\pi \mathrm{i}u_n(\mu)}v_n(\mu) + \frac{\mathrm{i}}{8\pi}\mathrm{e}^{-2\pi \mathrm{i}u_n(\mu)} + \frac{\mu}{4}\mathrm{e}^{-2\pi \mathrm{i}u_n(\mu)}v_n(\mu),\\
w_{n+1}(\mu) &= w_n(\mu) + \frac{\alpha''(\mu)}{2\pi} + 2\pi \mathrm{i}\mathrm{e}^{2\pi \mathrm{i}u_n(\mu)}v_n^2(\mu) + \mathrm{e}^{2\pi \mathrm{i}u_n(\mu)}w_n(\mu)\\
&\quad + \frac{1}{2}\mathrm{e}^{-2\pi \mathrm{i}u_n(\mu)}v_n(\mu) - \mu\frac{\pi \mathrm{i}}{2}\mathrm{e}^{-2\pi \mathrm{i}u_n(\mu)}v_n^2(\mu) + \frac{\mu}{4}\mathrm{e}^{-2\pi \mathrm{i}u_n(\mu)}w_n(\mu).
\end{aligned}$$

In particular, if we set $\mu = 0$, we have

$$\begin{aligned}
u_{n+1}(0) &= u_n(0) + \theta - \frac{\mathrm{i}}{2\pi}\mathrm{e}^{2\pi \mathrm{i}u_n(0)},\\
v_{n+1}(0) &= v_n(0) + \frac{\alpha'(0)}{2\pi} + \mathrm{e}^{2\pi \mathrm{i}u_n(0)}v_n(0) + \frac{\mathrm{i}}{8\pi}\mathrm{e}^{-2\pi \mathrm{i}u_n(0)},\\
w_{n+1}(0) &= w_n(0) + \frac{\alpha''(0)}{2\pi} + 2\pi \mathrm{i}\mathrm{e}^{2\pi \mathrm{i}u_n(0)}v_n^2(0) + \mathrm{e}^{2\pi \mathrm{i}u_n(0)}w_n(0) + \frac{1}{2}\mathrm{e}^{-2\pi \mathrm{i}u_n(0)}v_n(0).
\end{aligned} \tag{32}$$

By expanding (28) in power series we obtain the seeds of this iterative process,

$$u_0(0) = \frac{1}{2}, \quad v_0(0) = \frac{\mathrm{i}}{8\pi}, \quad w_0(0) = \frac{3\mathrm{i}}{32\pi}.$$

The only remaining question to apply recurrences (32) is to compute the Taylor expansion of the Arnold Tongue $\alpha = \alpha(\mu)$. In Appendix A we analytically compute the first terms of this expansion, obtaining

$$\alpha(0) = 2\pi\theta, \quad \alpha'(0) = \frac{\cos\pi\theta}{4\sin\pi\theta}, \quad \alpha''(0) = -\frac{3+\cos 4\pi\theta}{64(\sin\pi\theta)^2\sin 2\pi\theta}.$$

δ_0	$-0.1027942932921338 \pm 5 \times 10^{-9}$	$-0.1027942850211555 \pm 2 \times 10^{-16}$
δ_1	$-0.0044572205920056 \pm 2 \times 10^{-13}$	$-0.0044572205922061 \pm 2 \times 10^{-15}$
δ_2	$-0.0015246277775224 \pm 6 \times 10^{-13}$	$-0.0015246277774431 \pm 2 \times 10^{-15}$

Table 1: Numerical values of the Taylor coefficients of $\Delta(\mu)$ in (29) and estimated errors using the direct method (20) (left) and the extrapolation method of third order induced by (38) (right).

Remark 7. *The computation of the Taylor expansion of $\alpha(\mu)$ is usually done to fulfill the necessary conditions for the map $f_\mu(t)$ in (27) to be conjugate to the rigid rotation $\mathcal{T}_\theta(t) = t + \theta$. Nevertheless, in Appendix A we use an analogous but slightly different way, which seems to lead to easier computations. Precisely, we ask $f_\mu(t)$ to be conjugate to the semistandard map $g(t)$ of rotation number θ (see (30)). For explicit formulas for more coefficients see [12], where they are obtained combining numerical and semi-analytical ideas.*

In the first column of Table 1 we display the values of δ_0, δ_1 and δ_2 we have obtained after $N = F_{34}$ iterates and the behavior of the error. We estimate the numerical errors by comparing the values obtained for $N = F_{34}$ and $N = F_{33} = 5702887$.

3.3 Improvement of the results

Once we have computed the Taylor polynomial (29), our next step is to compare this truncated Taylor expansion with the function $\Delta(\varepsilon)$ in order to check their agreement as a function of ε. At the present moment, one could compare the Taylor polynomial with the value of $\Delta(\varepsilon)$ obtained through formula (25). However, in order to avoid numerical errors produced by subtracting two very similar quantities, we want to improve the numerical method (25), for the case of the Arnold family, to get a more accurate value for $\Delta(\varepsilon)$.

Our first guess is to try to extend the ideas introduced in [15], for computing rotation numbers of circle maps with high precision, in order to accelerate the convergence speed of formula (19). Unfortunately, this methodology fails in the present context. However, let us explain briefly the main points of this approach, adapted to the problem at hand, because the reasons of this failure help us in order to motivate the subsequent approach.

We look at formula (18) and introduce the sequence $\{S_N^j\}$ of "iterated sums" of S_N. Concretely, we set $S_N^1 = S_N$ and define inductively

$$S_N^j = \sum_{k=0}^{N-1} S_k^{j-1}, \quad j \geq 2.$$

The idea of [15] consists in taking a fixed j and to identify the asymptotic behavior of S_N^j, as $N \to +\infty$. This behavior is used to improve the convergence speed of $\hat{\xi}_0 = \lim_{N\to+\infty} S_N/N$, and hence of Δ, by means of an extrapolation process. The motivation behind this approach is that by increasing j one hopes to obtain faster convergence. Here we discuss only what happens for S_N^2.

From formula (18), we obtain

$$S_N^2 = \frac{(N-1)N}{2}\hat{\xi}_0 + NA + \sum_{k\in\mathbb{Z}} \hat{\xi}_k \frac{1 - \mathrm{e}^{2\pi \mathrm{i} kN\theta}}{(1 - \mathrm{e}^{2\pi \mathrm{i} k\theta})^2}, \tag{33}$$

where $A = \sum_{k\in\mathbb{Z}} \hat{\xi}_k/(1 - \mathrm{e}^{2\pi\mathrm{i}k\theta})$ is independent of N. This formula suggests to ask for the validity of the following asymptotic expression:

$$\hat{S}^2_N = \frac{2}{(N-1)N} S^2_N = \hat{\xi}_0 + \frac{2}{N} A + O\left(\frac{1}{N^2}\right). \tag{34}$$

Assuming (34) to be true, one can use Richardson's extrapolation to improve the computation of $\hat{\xi}_0$. For instance:

$$\hat{\xi}_0 = 2\hat{S}^2_{2N} - \hat{S}^2_N + O\left(\frac{1}{N^2}\right). \tag{35}$$

Unfortunately, we cannot expect the asymptotic expression (34) to hold for any N, and hence (35) does not provide better results than (19).

To motivate this assertion, we observe that formula (17) is obtained evaluating $\xi(x)$ (see (16)) at the point $x = a - \mathrm{i}\Delta$, which is on the boundary of its domain of analyticity (3). Then, its Fourier coefficients ξ_k get multiplied by $\mathrm{e}^{2\pi\mathrm{i}k(a-\mathrm{i}\Delta)}$. So, if $k > 0$, the modified coefficients $\hat{\xi}_k = \xi_k \mathrm{e}^{2\pi k\Delta}\mathrm{e}^{2\pi i k a}$ are no longer exponentially small decreasing (recall that $|\xi_k| \sim O(\mathrm{e}^{-2\pi|k|\Delta})$). We can expect, at most, the function ξ being Hölder continuous at the boundary (see [4]), so the coefficients $|\hat{\xi}_k| \sim O(k^{-\nu})$ when $k \to +\infty$, with $0 < \nu \le 1$. Then, as the sum (33) contains the (small) divisors $(1 - \mathrm{e}^{2\pi\mathrm{i}k\theta})^2$, we cannot expect the $O(1/N^2)$ behavior of the remainder in (34) to be uniform as function of N. In fact, numerical experiments suggest that this remainder is still of order $O(1/N)$.

Thereby, we have to look for a different method in order to improve (19), using N of moderate size. For this purpose, we focus on the assertion (15) of Proposition 3. This formula suggests that, if we pick up the sequence of values of N given by the denominators $\{q_j\}_{j\ge 0}$ of the convergents of θ, then we can ask if $S_{q_j}/q_j - \hat{\xi}_0$ behaves in a certain controlled form.

Now we are going to discuss this behavior for the case of the standard map and for values of the parameters taken on its Arnold Tongue when the rotation number θ is the Golden Mean. In Section 3.5 we will discuss more briefly other quadratic irrational rotation numbers. All the results we present are based on numerical experiments.

For the Golden Mean, it is well-known that the sequence of its convergents is given by $\{F_j/F_{j+1}\}_{j\ge 0}$, where $\{F_j\}_{j\ge 0}$ are the so-called *Fibonacci numbers*, defined as

$$F_0 = 1, \qquad F_1 = 1, \qquad F_{j+1} = F_j + F_{j-1}, \quad j \ge 1.$$

Then, we formulate the following conjecture for the sums S_{F_j}.

Conjecture 8. *Let θ be the Golden Mean and $S_{F_n} = S_{F_n}(\varepsilon)$, $n \ge 1$, be the sums defined in (18) for the scaled standard map* (23)*, when $\alpha = \alpha(\varepsilon)$ is taken on its Arnold Tongue of rotation number θ (see also* (27)*). Then, given a fixed $0 \le \varepsilon < 1$, there exist constants $\gamma, \delta \in \mathbb{R}$ such that*

$$\frac{1}{F_n} S_{F_n} = a - \mathrm{i}\Delta + \xi_0 + \frac{\mathrm{i}\gamma + (-1)^{\mathrm{n}}\delta}{F_n} + o\left(\frac{1}{F_n}\right), \tag{36}$$

where $\lim_{n\to+\infty} F_n \cdot o(1/F_n) = 0$.

By assuming this property true, we can improve the approximation for $\Delta = \Delta(\varepsilon)$ provided by (19) by extrapolation from two consecutive Fibonacci numbers:

$$\Delta \approx -\frac{\mathrm{Im}(S_{F_n} - S_{F_{n-1}})}{F_{n-2}}. \tag{37}$$

We point out that Conjecture 8 means that there is an asymptotic line for the iterates $\mathrm{Im}(S_{F_n})$, as function of $1/F_n$, given by $y = -\Delta x + \gamma$. In case we were dealing with a smooth function, the existence of this asymptotic line is not surprising in any sense, and it is usual in many (computational) contexts (rotation numbers, Lyapunov exponents, ...). However, we stress this property in our case, because as we are working on the boundary of the domain of analyticity of the conjugation, in principle we can only expect the map ξ to be (Hölder) continuous, but not differentiable, on the boundary. Thus, we are not convinced that the existence of this asymptotic behavior is as natural as it is for the smooth case. For instance, Conjecture 8 is not true if we consider the sums S_N for arbitrary values of N. Moreover, we observe that there is not properly an asymptotic line for the sequence defined by $\mathrm{Re}(S_{F_n})$, because it oscillates, depending of n being even or odd, between the lines $y = (a + \xi_0)x \pm \delta$.

In Section 3.5 we show that the asymptotic behavior of S_N depends strongly of the arithmetic properties of the rotation number θ.

We do not plan to prove Conjecture 8, but in Appendix B we use semi-analytic ideas to relate it with the asymptotic behavior of the Fourier coefficients of ξ, which is discussed, also numerically, in Section 3.4.

Still working with the Arnold family and the Golden Mean, one can think now about the possibility of refining Conjecture 8 and of fitting the terms $o(1/F_n)$, in order to obtain the asymptotic expansion of S_{F_n}/F_n as a function of F_n. More specifically, as we are only interested on its imaginary part, we state the following extension of Conjecture 8, also based on strong numerical evidences.

Conjecture 9. *With the same notations of Conjecture 8. Given a fixed* $0 \le \varepsilon < 1$, *there exist constants* $\gamma, A_i^{(\pm 1)} \in \mathbb{R}$, *for* $i \ge 1$, *such that*

$$\mathrm{Im}\left(\frac{S_{F_n}}{F_n}\right) = -\Delta + \frac{\gamma}{F_n} + \frac{A_1^{(-1)^n}}{F_n^{1+\theta}} + \frac{A_2^{(-1)^n}}{F_n^{2}} + \frac{A_3^{(-1)^n}}{F_n^{2+\theta}} + \frac{A_4^{(-1)^n}}{F_n^{3}} + \cdots . \tag{38}$$

In formula (38) we use the expressions $A_i^{(-1)^n}$ to denote the fact that these coefficients are different for n being even or odd. Moreover, we also stress that some exponents of this asymptotic expansion are non-integers but related with the rotation number, $\theta = (\sqrt{5} - 1)/2$.

If we assume the validity of (38), then we have the chance of applying some steps of a generalized "Richardson's extrapolation" to it, to improve the numerical computation of Δ.

Remark 10. *In order to denote the different methods for extrapolating* Δ *from formula* (38), *we introduce the following notation. After computing up to* F_n *iterates of the critical point, we call* zero order extrapolation *the approximation* $\Delta \approx -\mathrm{Im}(S_{F_n}/F_n)$ *(see* (19)*). We call* first order extrapolation *to formula* (37)*. The* second order extrapolation *is defined by considering in formula* (38) *the corrections up to order* $1/F_n^{1+\theta}$. *Then, taking into account that* $A_1^{(+1)}$ *and* $A_1^{(-1)}$ *take different values, we have to compute the sum* S_N *for four consecutive Fibonacci numbers,* $N \in \{F_{n-3}, F_{n-2}, F_{n-1}, F_n\}$, *and to solve a* 4*-dimensional linear system in order to extrapolate* Δ *with an error of* $O(1/F_n^2)$. *Analogously, we construct the higher order extrapolation methods.*

In Figure 2 we give the comparison between the results obtained using extrapolation of order from zero to three, according to the previous remark.

On the left we plot the errors for these four different extrapolation methods as a function of ε. For this purpose, we have computed up to $N = F_{34}$ iterates of the critical point t_ε^* in (26) under the map (23), with $\alpha = \alpha(\varepsilon)$. Then we estimate the numerical error, as we did for δ_i in

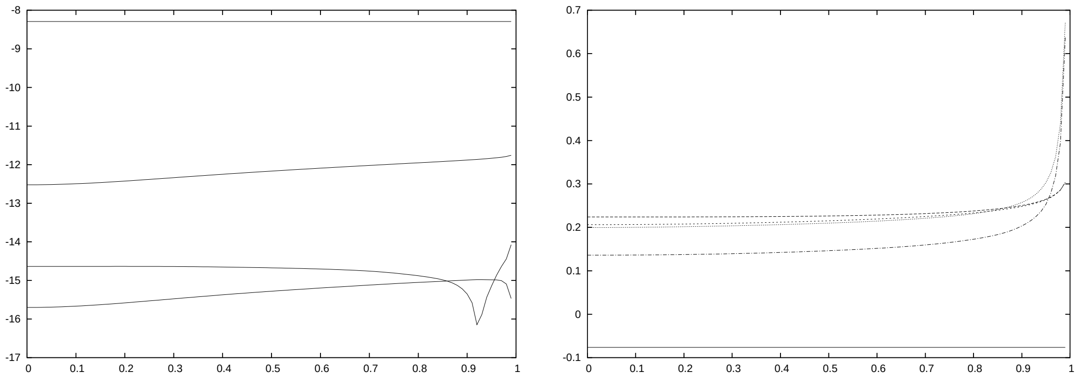

Figure 2: Left: $\log_{10}$ of the error in the numerical computation of $\Delta(\varepsilon)$ versus ε for $\theta = (\sqrt{5}-1)/2$. We use formula (19) and the first three extrapolation methods using (38). Right: the coefficients γ, $A_1^{(\pm 1)}$, $A_2^{(\pm 1)}$ of the extrapolation versus ε.

Section 3.2, by comparing the values for $\Delta(\varepsilon)$ obtained with $N = F_{33}$ and $N = F_{34}$ iterates. As expected, the error curves decrease as a function of the extrapolation order, except when ε approaches to 1. When ε is close to 1 the precision of the computed Arnold Tongue is not enough to deal with high extrapolation orders. Conversely, when $\varepsilon = 0$, we are able to compute the Siegel radius of the Semistandard map (30) with 15 decimal digits using $N = F_{34}$ iterates.

On the right we plot the behavior of the coefficients γ (the lower one), $A_1^{(\pm 1)}$ and $A_2^{(\pm 1)}$ as a function of ε. Let us observe that the numerical values of these coefficients show that, even though Δ varies with ε, they remain almost constant when ε is close to zero. Moreover, we also note that even though $A_i^{(+1)}$ and $A_i^{(-1)}$ are very similar, for a fixed ε, we cannot achieve the precision for $\Delta(\varepsilon)$ displayed in the left plot if do not take into account their difference in the expansion (38).

Figure 3 shows the same quantities as Figure 2, on the left the errors in the computation of Δ for the different extrapolation methods and on the right the coefficients of formula (38), but now for a fixed value of $\varepsilon = 0.1$ and different values of $N = F_n$, up to $n = 34$. For a better understanding of the plots we joint separate points with lines.

One can think about the possibility that the derivatives of Δ with respect to ε also verify a formula analogous to (38). Taking into account that formula (38) depends on ε, Figure 2 suggests that the coefficients γ and $A_i^{(\pm 1)}$ are smooth functions of ε. If this were the case, one could apply the extrapolation process to compute the coefficients δ_0, δ_1, δ_2 of the Taylor series of $\Delta(\varepsilon)$ in (29).

In the last column of Table 1 we give the improved coefficients δ_0, δ_1 and δ_2 after performing extrapolation of order three and $N = F_{34}$ iterates. We notice that there is not a major improvement in the computation of δ_1 and δ_2 with respect to the previous one. One possible explanation is the almost constant behavior of the coefficients γ, $A_1^{(\pm 1)}$ and $A_2^{(\pm 1)}$ for small ε, so that their derivatives are close to zero. Then, if we differentiate formula (38), the derivatives of the correction coefficients γ, $A_1^{(\pm 1)}$, $A_2^{(\pm 1)}$ almost vanish. This means that the direct computation of δ_i, $i = 1, 2$ using (20) has an error smaller than expected *a priori*, which makes the extrapolation almost useless.

Left plot in Figure 4 shows this phenomenon for δ_2. We plot the errors for the different

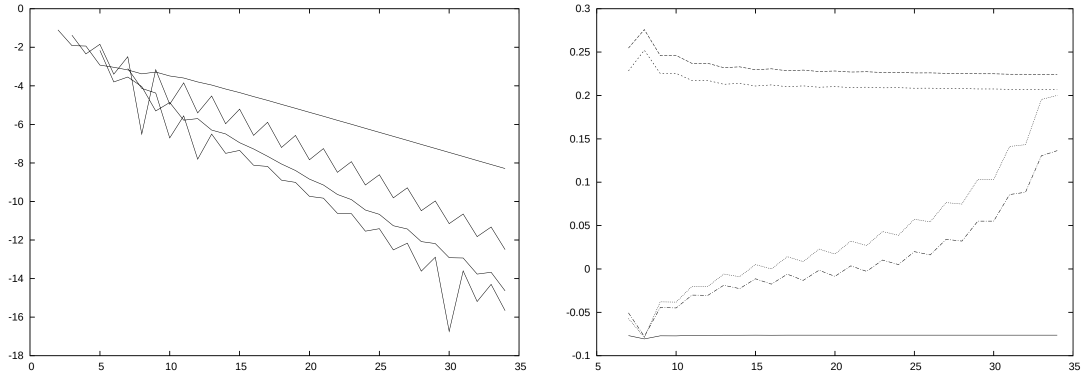

Figure 3: Left: $\log_{10}$ of the error in the numerical computation of $\Delta(0.1)$ by using the same methods as in Figure 2, versus the index n of the Fibonacci F_n corresponding to the number of iterates used. Right: the coefficients γ, $A_1^{(\pm 1)}$, $A_2^{(\pm 1)}$ of the extrapolation versus n.

methods used: direct computation (20) and extrapolation of order 1, 2 and 3.

Right plot in Figure 4 shows the agreement between the extrapolated value of $\Delta(\varepsilon)$ and the different Taylor approximations: the constant one δ_0, the lineal approximation $\delta_0 + \varepsilon^2\delta_1$ and the quadratic approximation $\delta_0 + \varepsilon^2\delta_1 + \varepsilon^4\delta_2$, as a function of ε. The plot shows the numerical error between $\Delta(\varepsilon)$ and these three different approximations. All the quantities are computed using the extrapolation method of order 3 with $N = F_{34}$ iterates.

3.4 Computation and asymptotics of the Fourier coefficients

In this section we consider the function η_ε of (24), giving the conjugation to a rotation of the scaled Arnold family (23) when $\alpha = \alpha(\varepsilon)$. Our goal is to compute (numerically) some Fourier coefficients of the periodic part of η_ε, when the rotation number θ is the Golden Mean (see (17)). Next to that, we identify the asymptotic behavior of these coefficients (see Conjecture 13). For our purposes, the most interesting point referring to this behavior is that we can establish a natural connection between Conjecture 13 and Conjecture 8. This connection is discussed in Appendix B.

To compute the Fourier coefficients of η_ε, namely $\hat{\xi}_k = \hat{\xi}_k(\varepsilon)$, we use the method introduced in Section 2.3. We fix the value of $\varepsilon \in [0, 1)$ and, for a given $k \geq 0$, we consider the modified sums S_N^k of (21) and the limit (22) (recall that coefficients with $k < 0$ are exponentially small in k). Then, numerical experiments suggest a similar behavior as (36) for these sums when N is a Fibonacci number. Concretely,

$$\frac{S_{F_n}^k}{F_n} = \hat{\xi}_k + \frac{B_k^{(-1)^n}}{F_n} + o\left(\frac{1}{F_n}\right),$$

where $B_k^{(\pm 1)} = B_k^{(\pm 1)}(\varepsilon)$ are complex numbers (compare also with (38)). So, analogously we did with Δ in Section 3.3, we can try to improve the computation of $\hat{\xi}_k$ by applying a Richardson-like extrapolation to this formula. The numerical results show that this methodology to refine these Fourier coefficients works quite well up to “moderate values of k”, without looking for higher order asymptotics like (38), and is enough for our purposes. For instance, we have not observed

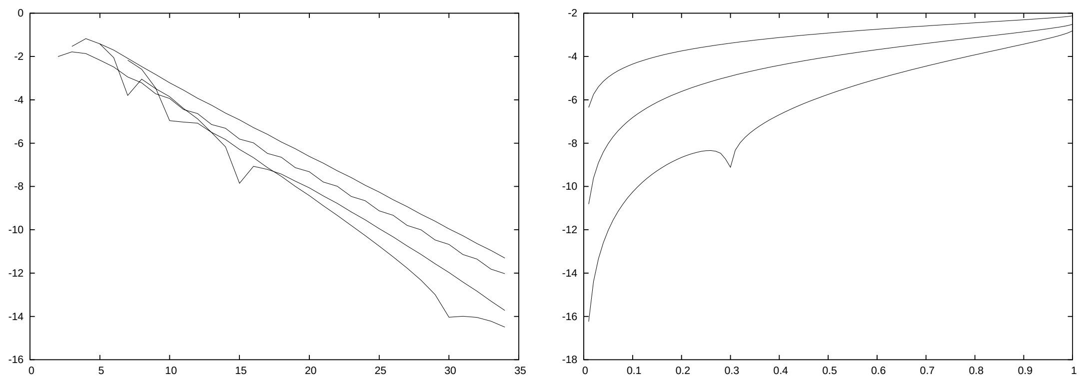

Figure 4: Left: $\log_{10}$ of the error in the numerical computation of δ_2 as function of the index n of the Fibonacci number F_n, using extrapolation of order from zero to three. Right: $\log_{10}$ of the error in the adjustment of $\Delta(\varepsilon)$ by its constant, linear or quadratic Taylor polynomial.

any problem to compute them for $0 \leq \varepsilon \leq 0.5$ and $0 \leq k \leq 1600$, with the precision displayed in the last column of Table 2.

To discuss the asymptotics of these Fourier coefficients, we first introduce the following property of the Fibonacci numbers. See [15] for the proof.

Lemma 11. *The set of Fibonacci numbers* $\{F_n\}_{n\geq 1}$ *is a* complete set *of integer numbers. More precisely, every* $m \in \mathbb{N}$ *admits a unique decomposition as sum of non-consecutive Fibonacci numbers. It means that there is a unique correspondence* $m \mapsto \{j_1, \ldots, j_{s(m)}\} \subset \mathbb{N}$*, where* $j_1, \ldots, j_{s(m)}$ *are non-consecutive integers, with* $s(m) \geq 1$*, and* $m = F_{j_1} + \cdots + F_{j_{s(m)}}$.

We observe that from Lemma 11 we can define the following relation of equivalence in $\mathbb{N}$.

Definition 12. *Given* $m, m' \in \mathbb{N}$*, we say that they belong to the same class of* generalized Fibonacci numbers*,* $\mathcal{F}(m) = \mathcal{F}(m')$*, if* $s \equiv s(m) = s(m')$ *and* $j_1 - j_1' = \cdots = j_s - j_s'$.

Thus, the Fibonacci numbers themselves define a class of equivalence, $\mathcal{F}(F_n)$. For instance, the sum of two Fibonacci numbers of the form $F_n + F_{n+2}$ defines a different class. To label these classes, we introduce an order between them. We say that $\mathcal{F}(m) < \mathcal{F}(m')$ if $\min\{q \in \mathcal{F}(m)\} < \min\{q \in \mathcal{F}(m')\}$. Then, the label $j \in \mathbb{N}$ of $\mathcal{F}(m)$ is defined by its order position in the set of classes. For instance, the label of the set of Fibonacci numbers is 1. The label of the class $\mathcal{F}(4) = \mathcal{F}(F_n + F_{n+2})$ is 2. We denote the elements of the j-class as $\mathcal{F}^j = \{F_1^j < F_2^j < \cdots\}$.

We observe that the elements of $\mathcal{F}^j$ verify the same recurrence as the Fibonacci numbers, $F_{n+1}^j = F_n^j + F_{n-1}^j$. As a consequence, they can be expressed in the following form,

$$F_n^j = A^j \left(\frac{1}{\theta}\right)^n + B^j (-\theta)^n, \qquad n \geq 1, \tag{39}$$

for certain constants A^j and B^j. For instance, $A^1 = 1/(1+\theta^2)$ and $B^1 = \theta^2/(1+\theta^2)$. Here we give the first six classes of generalized Fibonacci numbers and their generators.

$\mathcal{F}^1 = \mathcal{F}(F_n) = \{1, 2, 3, \ldots\}$, $\quad \mathcal{F}^4 = \mathcal{F}(F_n + F_{n+4}) = \{9, 15, 24, \ldots\}$,
$\mathcal{F}^2 = \mathcal{F}(F_n + F_{n+2}) = \{4, 7, 11, \ldots\}$, $\quad \mathcal{F}^5 = \mathcal{F}(F_n + F_{n+2} + F_{n+4}) = \{12, 20, 32, \ldots\}$,
$\mathcal{F}^3 = \mathcal{F}(F_n + F_{n+3}) = \{6, 10, 16, \ldots\}$, $\quad \mathcal{F}^6 = \mathcal{F}(F_n + F_{n+5}) = \{14, 23, 37, \ldots\}$.

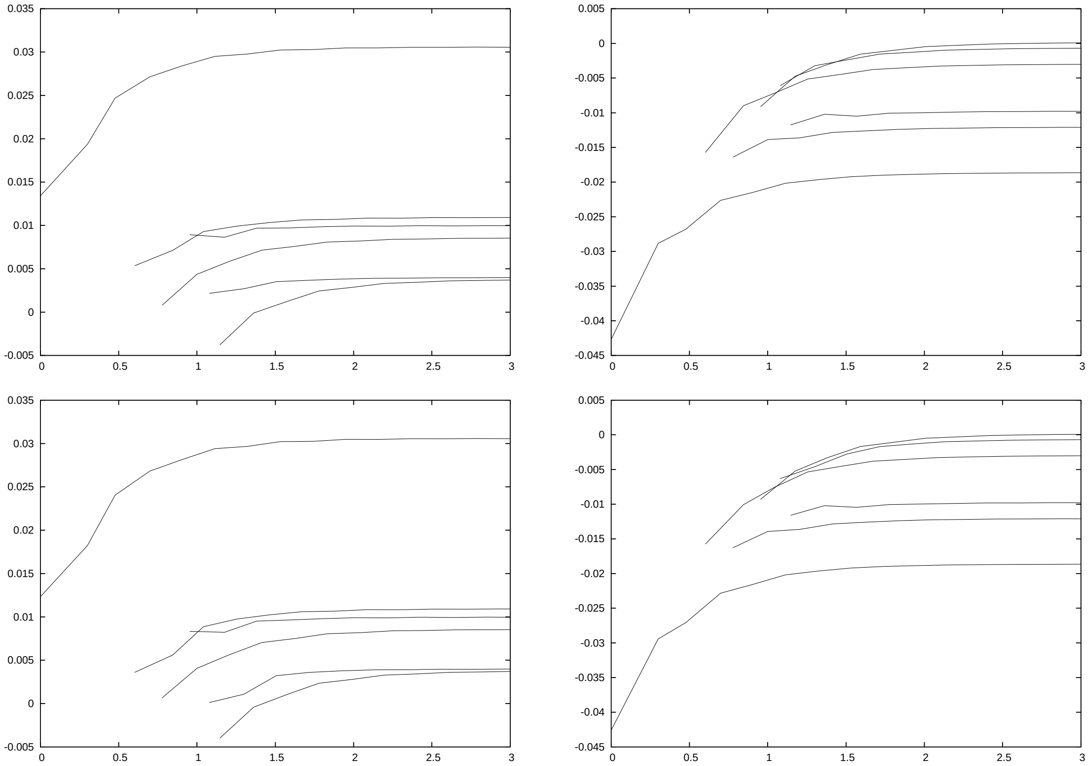

Figure 5: The two plots on the top correspond to $\varepsilon = 0$ and those on the bottom to $\varepsilon = 0.5$. Left: $(-1)^n F_n^j \cdot \mathrm{Re}(\hat{\xi}_{F_n^j})$ versus $\log_{10} F_n^j$, $j = 1, \cdots, 6$. Right: $F_n^j \cdot \mathrm{Im}(\hat{\xi}_{F_n^j})$ versus $\log_{10} F_n^j$, $j = 1, \cdots, 6$.

To finish this review of properties of the generalized Fibonacci numbers we observe that, for a given $m \in \mathbb{N}$, it is easy to control its associated small divisor, defined as $\min_{k\in\mathbb{Z}}\{|m\theta - k|\}$, if we know which family m belongs to. Thus, if $m = F_n^j$, this minimum is achieved when $k = F_{n-1}^j$, so that

$$F_n^j\theta - F_{n-1}^j = -(-\theta)^{n-1}(1+\theta^2)B^j = (-1)^n \frac{1+\theta^2}{\theta} A^j B^j \frac{1}{F_n^j} + O\left(\frac{1}{(F_n^j)^3}\right). \tag{40}$$

Numerical experiments with the modified Fourier coefficients $\hat{\xi}_k(\varepsilon)$ suggest the following behavior (see Figure 5).

Conjecture 13. *Given a fixed* $0 \le \varepsilon < 1$*, for any class of generalized Fibonacci numbers* $\mathcal{F}^j$*, there exist real constants* c^j *and* d^j *such that*

$$\hat{\xi}_{F_l^j} = \frac{(-1)^l c^j + \mathrm{i} d^j}{F_l^j} + o\left(\frac{1}{F_l^j}\right), \quad l \to \infty.$$

In Figure 5 we illustrate Conjecture 13 for two different values, $\varepsilon = 0$ and $\varepsilon = 0.5$, and the six first families of generalized Fibonacci numbers. We observe that the value of $k \cdot \mathrm{Im}(\hat{\xi}_k)$ has different (clearly defined) limit depending on which family the index k belongs to. Similar behavior is observed for $k \cdot \mathrm{Re}(\hat{\xi}_k)$, taking into account the oscillation pointed out in Conjecture 13. We also remark that the scaled Fourier coefficients $\hat{\xi}_k(\varepsilon)$ change very slowly as functions of ε .

$\mathcal{F}^j$	n	F_n^j	$F_n^j \times \hat{\xi}_{F_n^j}$	error of $F_n^j \times \hat{\xi}_{F_n^j}$
$\mathcal{F}^1 = \mathcal{F}(1)$	16	1597	$(-1)^n \times 0.03052 - 0.01867\,\mathrm{i}$	$5 \times 10^{-5} + 5 \times 10^{-5}\,\mathrm{i}$
$\mathcal{F}^2 = \mathcal{F}(4)$	13	1364	$(-1)^n \times 0.01091 - 0.00301\,\mathrm{i}$	$3 \times 10^{-5} + 6 \times 10^{-6}\,\mathrm{i}$
$\mathcal{F}^3 = \mathcal{F}(6)$	12	1220	$(-1)^n \times 0.00854 - 0.01211\,\mathrm{i}$	$3 \times 10^{-5} + 5 \times 10^{-6}\,\mathrm{i}$
$\mathcal{F}^4 = \mathcal{F}(9)$	11	1131	$(-1)^n \times 0.00996 + 0.00009\,\mathrm{i}$	$9 \times 10^{-6} + 2 \times 10^{-6}\,\mathrm{i}$
$\mathcal{F}^5 = \mathcal{F}(12)$	11	1508	$(-1)^n \times 0.00398 - 0.00068\,\mathrm{i}$	$1 \times 10^{-5} + 2 \times 10^{-6}\,\mathrm{i}$
$\mathcal{F}^6 = \mathcal{F}(14)$	10	1076	$(-1)^n \times 0.00372 - 0.00979\,\mathrm{i}$	$6 \times 10^{-6} + 1 \times 10^{-6}\,\mathrm{i}$

Table 2: Numerical values of the asymptotic coefficients displayed in Figure 5 for $\varepsilon = 0$ and estimated errors in their computation. With the factor $(-1)^n$ we want to stress the oscillatory character of the real part with respect to n.

In Table 2 we give the numerical values of the asymptotic coefficients c^j, d^j for $\varepsilon = 0$ and the six families considered above. This is done by computing $F_n^j \cdot \hat{\xi}_{F_n^j}$ for a big value of F_n^j. See Table 2 for more details. The error in the computation of these coefficients has been estimated analogously as we did in the previous sections. In order to identify the different families in Figure 5, we observe that the asymptotic coefficients verify:

$$c_1 > c_2 > c_4 > c_3 > c_5 > c_6, \quad d_1 > d_3 > d_6 > d_2 > d_5 > d_4.$$

3.5 Other rotation numbers

Albeit this is not the main objective of this work, in this section we investigate the validity of Conjecture 8 for general rotation numbers.

Given a rotation number θ, we denote by $\theta = [0; a_1, a_2, \cdots]$ its continuous fraction expansion and by $\{p_n/q_n\}_{n\geq 0}$ its convergents. For this rotation number we consider the sums S_{q_n} of (18) for the semistandard map (30) as done in (31). We have studied numerically the asymptotics of the imaginary part of these sums as a function of q_n and we have observed the following behavior.

Conjecture 14. *With the notations above, we have:*

- *If the continuous fraction expansion of θ is constant, $\theta = [0; a, a, \cdots]$, Conjecture 8 holds:*

$$\mathrm{Im}\left(\frac{S_{q_n}}{q_n}\right) = -\Delta + \frac{\gamma}{q_n} + o\left(\frac{1}{q_n}\right), \tag{41}$$

  where γ is independent of n. The same behavior is observed if the coefficients a_n of the continuous fraction expansion of θ become constant for $n \geq n_0$.

- *For the rest of quadratic irrationals formula (41) is not true anymore. Nevertheless, if k is the period of the continuous fraction expansion of θ, we have detected the following generalization of (41):*

$$\mathrm{Im}\left(\frac{S_{q_n}}{q_n}\right) = -\Delta + \frac{\gamma(n)}{q_n} + o\left(\frac{1}{q_n}\right), \tag{42}$$

  where $\gamma(n)$ is a k-periodic function, $\gamma(n+k) = \gamma(n)$, $\forall n \in \mathbb{N}$.

$\theta = \sqrt{2} - 1 = [0; 2, 2, 2, \ldots]$
$q_{18} = 6625109$
$\Delta = -0.103440721109 \pm 2 \times 10^{-8}$
$\Delta = -0.103440710968 \pm 7 \times 10^{-12}$
$\gamma = -0.06718 \pm 2 \times 10^{-5}$

$\theta = (\sqrt{13} - 3)/2 = [0; 3, 3, 3, \ldots]$
$q_{13} = 5097243$
$\Delta = -0.106852306065 \pm 3 \times 10^{-8}$
$\Delta = -0.168522946632 \pm 2 \times 10^{-11}$
$\gamma = -0.05812 \pm 4 \times 10^{-5}$

$\theta = \sqrt{3} - 1 = [0; 1, 2, 1, 2, \ldots]$
$q_{25} = 7865521$
$\Delta = -0.108305561281 \pm 9 \times 10^{-9}$
$\Delta = -0.108305554285 \pm 6 \times 10^{-12}$
$\gamma(0) = -0.05503 \pm 1 \times 10^{-5}$
$\gamma(1) = -0.09110 \pm 4 \times 10^{-5}$

$\theta = (\sqrt{3} - 1)/2 = [0; 2, 1, 2, 1, \ldots]$
$q_{24} = 5757961$
$\Delta = -0.104709458207 \pm 1 \times 10^{-8}$
$\Delta = -0.104709448652 \pm 9 \times 10^{-12}$
$\gamma(0) = -0.09109 \pm 4 \times 10^{-5}$
$\gamma(1) = -0.05502 \pm 2 \times 10^{-5}$

$\theta = (\sqrt{10} - 2)/2 = [0; 1, 1, 2, 1, 1, 2, \ldots]$
$q_{26} = 4052018$
$\Delta = -0.103540643256 \pm 2 \times 10^{-8}$
$\Delta = -0.103540627567 \pm 2 \times 10^{-11}$
$\gamma(0) = -0.07956 \pm 4 \times 10^{-5}$
$\gamma(1) = -0.07535 \pm 5 \times 10^{-5}$
$\gamma(2) = -0.06357 \pm 1 \times 10^{-5}$

$\theta = (\sqrt{17} - 3)/2 = [0; 1, 1, 3, 1, 1, 3, \ldots]$
$q_{23} = 4597824$
$\Delta = -0.106139432515 \pm 2 \times 10^{-8}$
$\Delta = -0.106139420833 \pm 2 \times 10^{-11}$
$\gamma(0) = -0.08051 \pm 5 \times 10^{-5}$
$\gamma(1) = -0.07054 \pm 6 \times 10^{-5}$
$\gamma(2) = -0.05371 \pm 1 \times 10^{-5}$

Table 3: Numerical examples of formula (42) for different quadratic irrational rotation numbers.

- *For general Diophantine rotation numbers, we have not observed any similar correlation between* $\mathrm{Im}(S_{q_n}/q_n)$ *and* $1/q_n$.

In Table 3 we illustrate Conjecture 14 for six different rotation numbers. For any number θ we give its continuous fraction expansion, the value of Δ computed using the direct formula (31) and the estimated error, the value of Δ and of the coefficients $\gamma(0), \cdots, \gamma(n-1)$ computed from (42) as well as their error. The total number of iterates q_n taken in any case is also displayed in the table.

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A Taylor expansion of the Arnold Tongue

In this section, we analytically compute the Taylor expansion, up to order ε^4, of the function $\alpha(\varepsilon)$ giving the parameterization of the Arnold Tongue T_θ for the Arnold family (1), for any Diophantine rotation number θ. As we know *a priori* that $\alpha(\varepsilon)$ is an analytic even function of ε, we use the notation introduced in Section 3.2. Then, we set $\mu = \varepsilon^2$, and we compute, in fact, the Taylor expansion of $\alpha(\mu)$, up to order μ^2, by working with the scaled family $f_\mu(t)$ defined in (27).

To compute $\alpha(\mu)$ we use that, being θ a Diophantine number, then f_μ is analytically conjugate to the rigid rotation $\mathcal{T}_\theta(t) = t+\theta$, for any $|\mu|<1$. But, instead of looking for an analytic conjugation between f_μ and $\mathcal{T}_\theta$, we look for a conjugation between f_μ and the semistandard map g of (30). We proceed in this way because we also know that g is analytically conjugate to $\mathcal{T}_\theta$. Thus, as g gives the limit when $\mu=0$ of the family f_μ (i. e., $g=f_0$), we expect the conjugation between both maps to take a simpler form than the one between f_μ and $\mathcal{T}_\theta$.

Hence, we look for $\alpha(\mu)$ and $\sigma_\mu(t)$ of the form

$$\alpha(\mu) = 2\pi\theta + \mu\alpha_1 + \mu^2\alpha_2 + \cdots, \qquad \sigma_\mu(t) = t + \mu\sigma_1(t) + \mu^2\sigma_2(t) + \cdots,$$

being $\sigma_i(t)$ periodic functions of period one, so that $f_\mu \circ \sigma_\mu(t) = \sigma_\mu \circ g(t)$. More concretely, we have to equate powers of μ of the following expressions:

$$\begin{aligned} f_\mu\circ\sigma_\mu(t) &= t+\mu\sigma_1(t)+\mu^2\sigma_2(t)+\cdots+\theta+\mu\frac{\alpha_1}{2\pi}+\mu^2\frac{\alpha_2}{2\pi}+\cdots-\frac{\mathrm{i}}{2\pi}\mathrm{e}^{2\pi\mathrm{i}(t+\mu\sigma_1(t)+\mu^2\sigma_2(t)+\cdots)} \\ &\quad +\mu\frac{\mathrm{i}}{8\pi}\mathrm{e}^{-2\pi\mathrm{i}(t+\mu\sigma_1(t)+\mu^2\sigma_2(t)+\cdots)}, \\ \sigma_\mu\circ g(t) &= t+\theta-\frac{\mathrm{i}}{2\pi}\mathrm{e}^{2\pi\mathrm{i}t}+\mu\sigma_1\left(t+\theta-\frac{\mathrm{i}}{2\pi}\mathrm{e}^{2\pi\mathrm{i}t}\right)+\mu^2\sigma_2\left(t+\theta-\frac{\mathrm{i}}{2\pi}\mathrm{e}^{2\pi\mathrm{i}t}\right)+\cdots. \end{aligned}$$

The order zero terms in μ are identical and the terms of order μ and μ^2 give the equations:

$$\sigma_1(t)+\frac{\alpha_1}{2\pi}+\mathrm{e}^{2\pi\mathrm{i}t}\sigma_1(t)+\frac{\mathrm{i}}{8\pi}\mathrm{e}^{-2\pi\mathrm{i}t} = \sigma_1\left(t+\theta-\frac{\mathrm{i}}{2\pi}\mathrm{e}^{2\pi\mathrm{i}t}\right), \tag{43}$$

$$\sigma_2(t)+\frac{\alpha_2}{2\pi}+\mathrm{e}^{2\pi\mathrm{i}t}\sigma_2(t)+\mathrm{i}\pi\mathrm{e}^{2\pi\mathrm{i}t}(\sigma_1(t))^2+\frac{1}{4}\mathrm{e}^{-2\pi\mathrm{i}t}\sigma_1(t) = \sigma_2\left(t+\theta-\frac{\mathrm{i}}{2\pi}\mathrm{e}^{2\pi\mathrm{i}t}\right). \tag{44}$$

One can easily realize that these equations have solutions for σ_1 and σ_2 taking the form:

$$\sigma_1(t) = A_{-1}\mathrm{e}^{-2\pi\mathrm{i}t}+\sum_{k\ge1}A_k\mathrm{e}^{2\pi\mathrm{i}kt}, \qquad \sigma_2(t) = B_{-2}\mathrm{e}^{-4\pi\mathrm{i}t}+B_{-1}\mathrm{e}^{-2\pi\mathrm{i}t}+\sum_{k\ge1}B_k\mathrm{e}^{2\pi\mathrm{i}kt}.$$

From equation (43) we derive the following conditions for α_1 and A_{-1},

$$A_{-1}+\frac{\mathrm{i}}{8\pi} = A_{-1}\mathrm{e}^{-2\pi\mathrm{i}\theta}, \qquad \frac{\alpha_1}{2\pi}+A_{-1} = -A_{-1}\mathrm{e}^{-2\pi\mathrm{i}\theta},$$

giving

$$A_{-1} = -\frac{\mathrm{e}^{\pi\mathrm{i}\theta}}{16\pi\sin\pi\theta}, \quad \alpha_1 = \frac{\cos\pi\theta}{4\sin\pi\theta}.$$

The remaining Fourier coefficients A_k can be computed recursively. For instance A_1 verifies

$$A_1 = \frac{1}{2}A_{-1}\mathrm{e}^{-2\pi\mathrm{i}\theta}+A_1\mathrm{e}^{2\pi\mathrm{i}\theta},$$

which gives

$$A_1 = -\mathrm{i}\frac{\mathrm{e}^{-2\pi\mathrm{i}\theta}}{64\pi(\sin\pi\theta)^2}.$$

From equation (44) we obtain the following conditions for B_{-2}, B_{-1} and α_2:

$$\begin{aligned} B_{-2}+\frac{1}{4}A_{-1} &= B_{-2}\mathrm{e}^{-4\pi\mathrm{i}\theta}, \\ B_{-1}+B_{-2}+\mathrm{i}\pi A_{-1}^2 &= -2B_{-2}\mathrm{e}^{-4\pi\mathrm{i}\theta}+B_{-1}\mathrm{e}^{-2\pi\mathrm{i}\theta}, \\ \frac{\alpha_2}{2\pi}+B_{-1}+\frac{1}{4}A_1 &= 2B_{-2}\mathrm{e}^{-4\pi\mathrm{i}\theta}-B_{-1}\mathrm{e}^{-2\pi\mathrm{i}\theta}, \end{aligned}$$

which give the values

$$B_{-2} = -\mathrm{i}\frac{\mathrm{e}^{3\pi \mathrm{i}\theta}}{128\pi \sin\pi\theta \sin 2\pi\theta}, \qquad B_{-1} = \frac{4+\mathrm{e}^{4\pi\mathrm{i}\theta} - \mathrm{e}^{2\pi\mathrm{i}\theta}}{512\pi(\sin\pi\theta)^2 \sin 2\pi\theta},$$

and finally,

$$\alpha_2 = \frac{-(3+\cos 4\pi\theta)}{128(\sin\pi\theta)^2\sin 2\pi\theta}.$$

B Motivation of Conjecture 8

The goal of this section is to show how Conjecture 8, referring to the asymptotic behavior when $n \to +\infty$ of the sums $S_{F_n}(\varepsilon)$ in (18), for the scaled standard map (23) and rotation number the Golden Mean, can be related with Conjecture 13, referring to the asymptotic behavior of the Fourier coefficients $\hat{\xi}_k(\varepsilon)$. The keystone of this connection is given by next result.

Lemma 15. *Let θ be the Golden Mean. Given any class of generalized Fibonacci numbers $\mathcal{F}^j$ (see Section 3.4) and arbitrary real numbers c^j and d^j, we consider the sums:*

$$M^j_{F_n} = \sum_{l\geq 1} \frac{(-1)^l c^j + \mathrm{i}d^j}{F^j_l} \frac{1-\mathrm{e}^{2\pi\mathrm{i}F^j_l F_n\theta}}{1-\mathrm{e}^{2\pi\mathrm{i}F^j_l\theta}}.$$

Then, there exist real values γ^j, δ^j such that

$$\lim_{n\to+\infty}\left(M^j_{F_n} - \left(\mathrm{i}\gamma^j + (-1)^n\delta^j\right)\right) = 0.$$

From this Lemma, one can guess how Conjecture 8 relies in Conjecture 13. First, we use that, for negative k, the corresponding Fourier coefficients $\hat{\xi}_k$ can be neglected because they are exponentially small. Second, for positive k, we use Conjecture 13. Moreover, to simplify the discussions, we assume that when $k > 0$ the coefficients $\hat{\xi}_k$ are *exactly* given by its asymptotic behavior, depending on which Fibonacci family k belongs to. With these two assumptions at hand, we consider the sum S_N of (18) for $N = F_n$. Then, using Lemma 15, the asymptotic behavior of S_{F_n}, when $n \to +\infty$, verifies

$$\begin{aligned}
\frac{1}{F_n}S_{F_n} &= \hat{\xi}_0 + \frac{1}{F_n}\sum_{k\in\mathbb{Z}\setminus\{0\}} \hat{\xi}_k \frac{1-\mathrm{e}^{2\pi\mathrm{i}kF_n\theta}}{1-\mathrm{e}^{2\pi\mathrm{i}k\theta}} \approx \hat{\xi}_0 + \frac{1}{F_n}\sum_{j\geq 1}\left(\sum_{l\geq 1}\frac{(-1)^l c^j + \mathrm{i}d^j}{F^j_l}\frac{1-\mathrm{e}^{2\pi\mathrm{i}F^j_l F_n\theta}}{1-\mathrm{e}^{2\pi\mathrm{i}F^j_l\theta}}\right) \\
&= \hat{\xi}_0 + \frac{1}{F_n}\sum_{j\geq 1} M^j_{F_n} \approx \hat{\xi}_0 + \frac{1}{F_n}\sum_{j\geq 1}\left(\mathrm{i}\gamma^j + (-1)^n\delta^j\right) = a - \mathrm{i}\Delta + \xi_0 + \frac{\mathrm{i}\gamma + (-1)^n\delta}{F_n},
\end{aligned}$$

if we assume the series $\gamma = \sum_{j\geq 1}\gamma^j$ and $\delta = \sum_{j\geq 1}\delta^j$ to be convergent.

Proof of Lemma 15: It is clear that the first question is the convergence of the sum $M^j_{F_n}$ itself, for any $n \geq 1$, but it follows immediately from the computations we are going to do.

Using (39), (40) and some straightforward computations, we have:

$$\begin{aligned}
F^j_l(1-\mathrm{e}^{2\pi\mathrm{i}F^j_l\theta}) &= F^j_l\left(1-\mathrm{e}^{2\pi\mathrm{i}(F^j_l\theta - F^j_{l-1})}\right) \\
&= F^j_l\left(1-\mathrm{e}^{2\pi\mathrm{i}(-1)^l(1+\theta^2)B^j\theta^{l-1}}\right) \\
&= (A^j\theta^{-l} + B^j(-\theta)^l)(2\pi\mathrm{i}(-1)^{l+1}(1+\theta^2)B^j\theta^{l-1} + O(\theta^{2l})) \\
&= 2\pi\mathrm{i}(-1)^{l+1}(1+\theta^2)\theta^{-1}A^jB^j + O(\theta^l). \qquad (45)
\end{aligned}$$

Here we stress that, when writing $O(\cdot)$ during the proof, it means that the coefficient controlling this order is independent of l and n.

To estimate the contribution of the term $1-\mathrm{e}^{2\pi \mathrm{i} F_l^j F_n\theta}$, we split the sum in two parts: for $l \le n$ and for $l \ge n+1$. Then, we use different approximations for this expression on any part. If $l \le n$ we have:

$$\begin{aligned} 1-\mathrm{e}^{2\pi \mathrm{i} F_l^j F_n\theta} &= 1-\mathrm{e}^{2\pi \mathrm{i} F_l^j (F_n\theta - F_{n-1})} = 1-\mathrm{e}^{2\pi \mathrm{i}\left[(-1)^n(1+\theta^2)B^1A^j\theta^{n-l-1}+O(\theta^{l+n})\right]} \\ &= 1-\mathrm{e}^{2\pi \mathrm{i}(-1)^n(1+\theta^2)B^1A^j\theta^{n-l-1}} + O(\theta^{l+n}). \end{aligned} \tag{46}$$

Moreover, if $l \ge n+1$ we have

$$1-\mathrm{e}^{2\pi \mathrm{i} F_l^j F_n\theta} = 1-\mathrm{e}^{2\pi \mathrm{i}(-1)^l(1+\theta^2)B^lA^1\theta^{l-n-1}} + O(\theta^{l+n}). \tag{47}$$

These computations motivate to introduce auxiliary sums $\widetilde{M}^j_{F_n}$, defined by taking the dominant terms of these expressions:

$$\begin{aligned} \widetilde{M}^j_{F_n} &= \sum_{l=1}^{n} \frac{(-1)^l c^j + \mathrm{i} d^j}{2\pi \mathrm{i}(-1)^{l+1}(1+\theta^2)\theta^{-1}A^jB^j}\left(1-\mathrm{e}^{2\pi \mathrm{i}(-1)^n(1+\theta^2)B^1A^j\theta^{n-l-1}}\right) \\ &\quad + \sum_{l\ge n+1} \frac{(-1)^l c^j + \mathrm{i} d^j}{2\pi \mathrm{i}(-1)^{l+1}(1+\theta^2)\theta^{-1}A^jB^j}\left(1-\mathrm{e}^{2\pi \mathrm{i}(-1)^l(1+\theta^2)B^lA^1\theta^{l-n-1}}\right) \\ &= \sum_{k=0}^{n-1} \frac{\mathrm{i}c^j + (-1)^{k+n+1}d^j}{2\pi(1+\theta^2)\theta^{-1}A^jB^j}\left(1-\mathrm{e}^{2\pi \mathrm{i}(-1)^n(1+\theta^2)B^1A^j\theta^{k-1}}\right) \\ &\quad + \sum_{k\ge 1} \frac{\mathrm{i}c^j + (-1)^{k+n+1}d^j}{2\pi(1+\theta^2)\theta^{-1}A^jB^j}\left(1-\mathrm{e}^{2\pi \mathrm{i}(-1)^{k+n}(1+\theta^2)B^lA^1\theta^{k-1}}\right) \\ &= \mathrm{i}\cdot \mathrm{Im}(\widehat{M}^j_{F_n}) + (-1)^n \mathrm{Re}(\widehat{M}^j_{F_n}), \end{aligned}$$

where

$$\begin{aligned} \widehat{M}^j_{F_n} &= \sum_{k=0}^{n-1} \frac{\mathrm{i}c^j + (-1)^{k+1}d^j}{2\pi(1+\theta^2)\theta^{-1}A^jB^j}\left(1-\mathrm{e}^{2\pi \mathrm{i}(1+\theta^2)B^1A^j\theta^{k-1}}\right) \\ &\quad + \sum_{k\ge 1} \frac{\mathrm{i}c^j + (-1)^{k+1}d^j}{2\pi(1+\theta^2)\theta^{-1}A^jB^j}\left(1-\mathrm{e}^{2\pi \mathrm{i}(-1)^k(1+\theta^2)B^lA^1\theta^{k-1}}\right). \end{aligned}$$

To obtain this expression, we have changed the indexes in the sums by $k = n-l$ when $l \le n$ and $k = l-n$ when $l \ge n+1$. Using the bound $|1-\mathrm{e}^{\mathrm{i}x}| \le |x|$, $\forall x \in \mathbb{R}$, it is clear that $\widehat{M}^j_{F_n}$ is finite for any n and that $\lim_{n\to+\infty} \widehat{M}^j_{F_n} = \mathrm{i}\gamma^j + \delta^j$ exists.

Finally, it only remains to control the difference between $M^j_{F_n}$ and $\widetilde{M}^j_{F_n}$. From the computations above and the orders of the remainders in (45), (46) and (47), it is clear that this error is controlled by a constant factor (independent of n) of the expression

$$\sum_{l=1}^{n} \theta^n + \sum_{l\ge n+1}(\theta^{l+n} + \theta^{2l-n-1}) = n\theta^n + \frac{\theta^{2n+1}}{1-\theta} + \frac{\theta^{n+1}}{1-\theta^2},$$

that goes to zero with n.

□