Maps

- A class of dynamical system where time is <u>discrete</u>, rather than continuous

Also known as <u>difference equations</u>, <u>iterated maps</u>, <u>recursion relations</u>

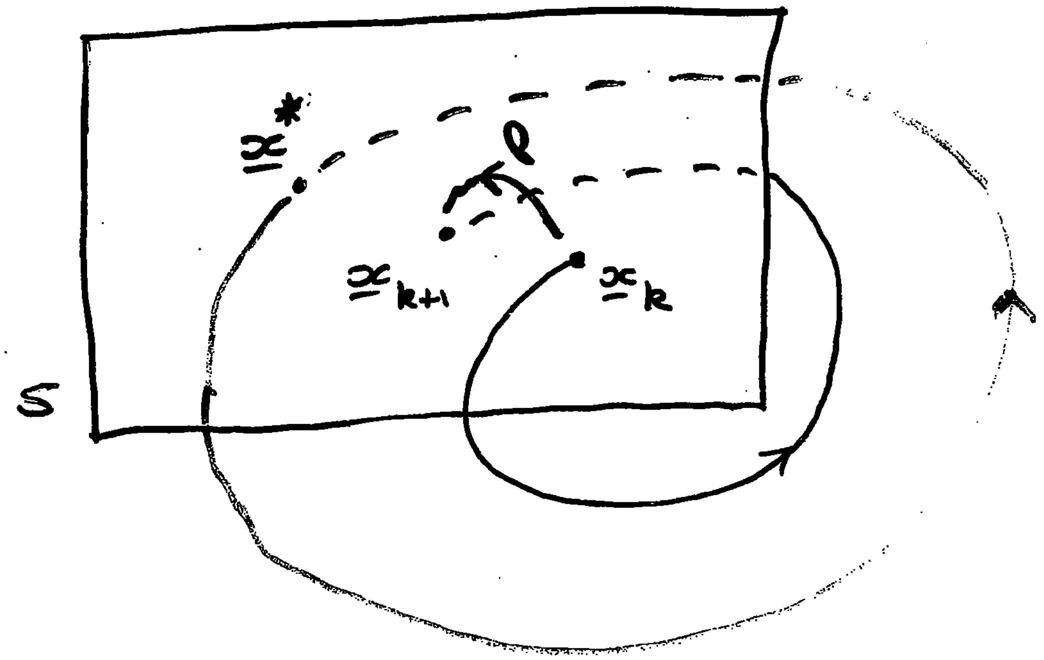
- Simple example: fix x_0, let
$x_1 = \cos x_0$, $x_2 = \cos x_1$,
$x_3 = \ldots$.

[Try it! What happens to x_n as $n \to \infty$?]

The rule $\boxed{x_{n+1} = \cos x_n}$ is a one-dimensional (1D) map.

The sequence $x_0, x_1, x_2, \ldots$ is the <u>orbit</u> starting from x_0.

- Maps arise naturally as tools for analyzing differential equations.



Let $\dot{\underline{x}} = \underline{f}(\underline{x})$ be an n-dimensional system

Let S be an $(n-1)$-dimensional surface of section, which is transversal to all trajectories (no trajectories parallel to S).

The Poincaré map $P: S \longrightarrow S$ is such that

$$\underline{x}_{k+1} = P(\underline{x}_k)$$

The fixed point $\underline{x}^*$ satisfies $\underline{x}^* = P(\underline{x}^*)$ corresponds to a closed orbit of $\dot{\underline{x}} = \underline{f}(\underline{x})$.

- Poincaré maps can turn difficult problems in differential equations into easier problems [but it is very difficult to get an explicit expression for P !]

$$\dot{x} + x = A \sin \omega t \qquad (\omega > 0)$$
$$x(0) = x_0$$

Solution $\quad x(t) = c_1 e^{-t} + c_2 \sin \omega t + c_3 \cos \omega t$

where $\quad x_0 = c_1 + c_3$

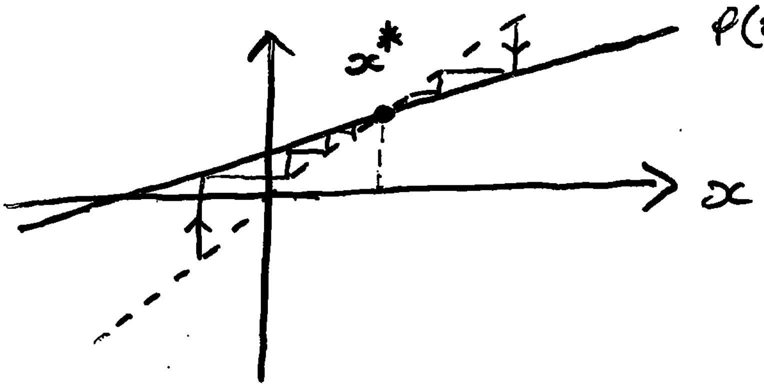
Let S be $\{ t = 0, \bmod 2\pi/\omega \}$

$\Rightarrow$ P is defined as

$$x_1 = P(x_0) = x(2\pi/\omega)$$

$$\Rightarrow \quad P(x_0) = x_0 e^{-2\pi/\omega} + c_3 \left(1 - e^{-2\pi/\omega}\right)$$

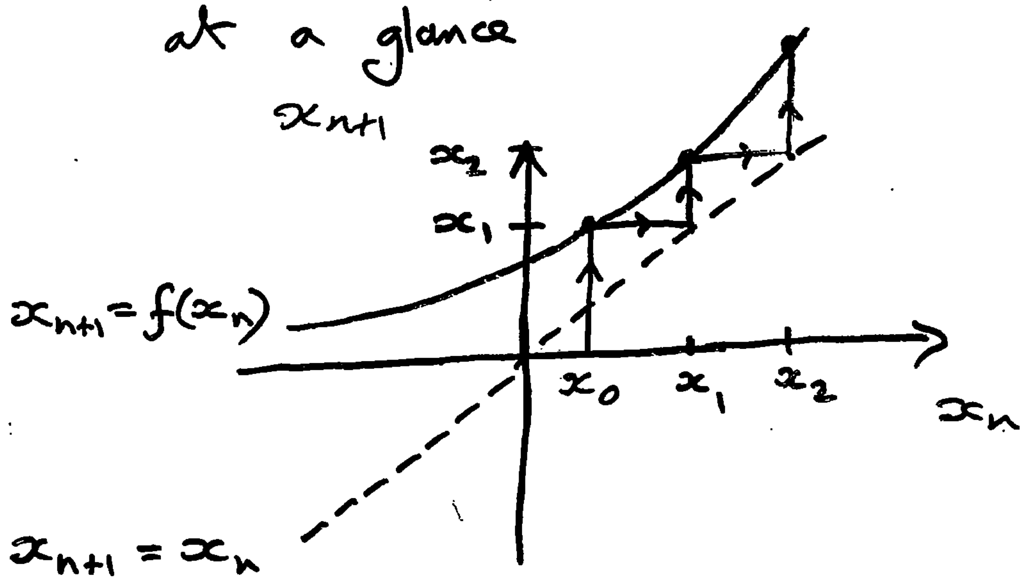
Plot $P(x)$ vs. x



No matter what the initial condition, x always ends up on same forced oscillation, <u>in this case</u>.

- **Cobweb** diagram (1D maps)

Allows us to see global behaviour at a glance



① Plot two lines $x_{n+1}=x_n$ and $x_{n+1}=f(x_n)$
[If they intersect we get fixed points of the map $x_{n+1}=x_n=f(x_n)=x^*=f(x^*)$]

② Choose initial condition x_0 & draw vertical line until it intersects the graph of f.

③ This gives output x_1. Now draw horizontal line until it intersects line $x_{n+1}=x_n$ and then move vertically until it intersects the graph of f.

④ This gives output x_2. Repeat step ③ as often as required.

⑤ Get orbit $x_0, x_1, x_2, \dots$

- Stability of fixed points (1D maps)

The map $x_{n+1} = f(x_n) = \cos x_n$

has a fixed point $x^* = \cos x^* = 0.739$

which is <u>stable</u> ($x_n \to x^*$ as $n \to \infty$).

For other maps, a fixed point can be <u>unstable</u>

$\boxed{x_{n+1} = x_n^2}$ has two fixed points

$x^* = 0$, $x^* = 1$. Set $x_0 = 0.1$, what happens to x_n as $x_n \to \infty$? Now set $x_0 = 1.1$, do the same. What happens?

There is a simple criterion for the 'stability' of fixed points of 1D maps. By stability, we mean <u>local stability</u> — that is, give the fixed point a small perturbation, and then ask what happens to that perturbation as $n \to \infty$. <u>Perturbation $\uparrow$ $\Rightarrow$ unstable,</u> <u>perturbation $\searrow 0$ $\Rightarrow$ stable.</u>

Suppose $x_{n+1} = f(x_n)$ has a fixed point $x_{n+1} = x_n = x^*$ such that $\underline{x^* = f(x^*)}$.

To determine the stability of x^*, look at a nearby orbit $x_n = x^* + \epsilon_n$ where $|\epsilon_n| \ll 1$ & see how ϵ_n behaves as $n \to \infty$.

So $$x_{n+1} = f(x_n)$$

$$\Rightarrow \quad x^* + \epsilon_{n+1} = f(x^* + \epsilon_n)$$

$$\underline{x^*} + \epsilon_{n+1} = \underline{f(x^*)} + f'(x^*)\epsilon_n + O(\epsilon_n^2)$$

$$\Rightarrow \quad \epsilon_{n+1} \approx f'(x^*)\,\epsilon_n \qquad \boxed{\text{linearised map}}$$

$$\Rightarrow \quad \epsilon_n = [f'(x^*)]^n \epsilon_0$$

So for $\epsilon_n \to 0$ as $n \to \infty$ we need that $\lambda \equiv f'(x^*)$, the <u>eigenvalue</u>, is such that

$$\boxed{|\lambda| = |f'(x^*)| < 1}$$

in which case the fixed point x^* is <u>linearly stable</u>

If $|\lambda| = |f'(x^*)| > 1 \Rightarrow$ unstable

If $|\lambda| = |f'(x^*)| = 1 \Rightarrow$ marginal case and need to do more work!

- Note the steps needed:

$$\boxed{x_{n+1} = f(x_n)}$$

Find fixed point(s) $\boxed{x^* = f(x^*)}$

Evaluate $f'(x)$ and then

find when $\boxed{-1 < f'(x^*) < 1}$

- Usually $f(\cdot)$ depends on parameters and varying <u>these</u> changes the eigenvalue, $\lambda = f'(x^*)$.

- When $f'(x^*) = 1$, have transcritical bifurcation

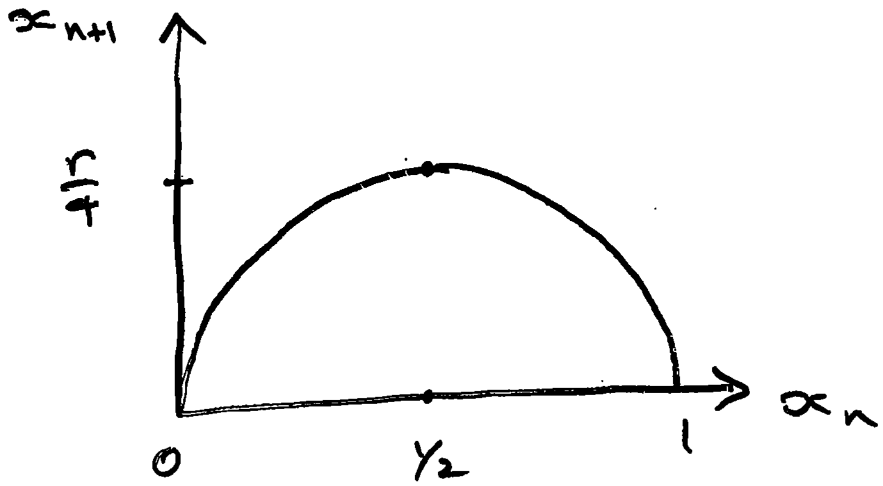
- When $f'(x^*) = -1$, have period doubling bifurcation (flip bifurcation)

- Maps are also used as models of natural phenomena in areas such as digital electronics, economics and population dynamics which gave the most famous example of a nonlinear map, the logistic map

$$x_{n+1} = r x_n (1 - x_n)$$

where $r \in [0, 4]$ so that $x_n \in [0, 1]$

[The map describes the competition, measured by r, between birth and death in a simple population]



- Exercises

  - Find both fixed points of the logistic map

    [$x^* = 0,\ 1 - 1/r$]

  - Over what range of r are they allowed?

    [$x^* = 0\ \forall r$, $x^* = 1 - 1/r$ needs $r \geq 1$]

  - What is the eigenvalue of the map?

    [$f'(x^*) = r - 2rx^*$]

  - For what range of r is $x^* = 0$ stable?

    [$r \in [0, 1)$]

  - For what range of r is $x^* = 1 - 1/r$ stable?

    [$r \in (1, 3)$]

  - What happens to iterates of the map in the range $r \in (3, 4]$?

    Chaos etc – not discussed here

- The logistic map has <u>m-cycles</u> (or <u>period m solutions</u>). For example a period 2 solution has

$$\left.\begin{array}{l} x_0 = x_2 = x_4 = \cdots = x^* \\ x_1 = x_3 = x_5 = \cdots = x^{**} \end{array}\right\} \; x^* \neq x^{**}$$

In other words, it has two points x^*, x^{**} such that

$$f(x^*) = x^{**} \quad , \quad f(x^{**}) = x^*$$

$$\Rightarrow \qquad f[f(x^*)] = x^*$$

[Usually write $f^2(x^*) = f[f(x^*)]$ so if f is quadratic, f^2 is quartic].

- The logistic map has a period 2 solutions '<u>born</u>' at $r = 3$, period 4 at $r = 3.449$, period 8 at $r = 3.54409$, ... and is chaotic in the range $r \in [3.569946, 4]$

- Local stability of fixed point of n-dimensional map

$$\underline{x}_{k+1} = P(\underline{x}_k)$$

where $\underline{x}$ is n-dimensional vector, with fixed point $\underline{x}^* = P(\underline{x}^*)$

Perturb – let $\underline{x}_k = \underline{x}^* + \underline{\epsilon}_k \Rightarrow$

$$\begin{aligned} \underline{x}^* + \underline{\epsilon}_{k+1} &= P(\underline{x}^* + \underline{\epsilon}_k) \\ &= P(\underline{x}^*) + [DP(\underline{x}^*)]\,\underline{\epsilon}_k + O(\|\underline{\epsilon}_k\|^2) \end{aligned}$$

where DP is Jacobian of P

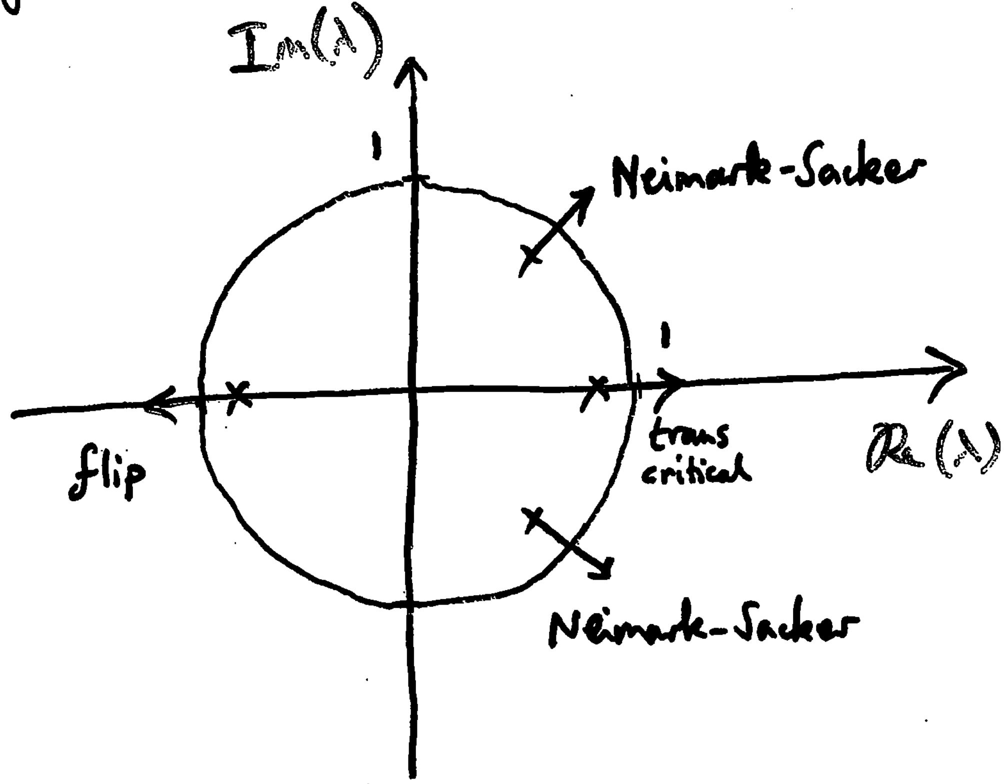
$$\Rightarrow \quad \underline{\epsilon}_{k+1} = [DP(\underline{x}^*)]\,\underline{\epsilon}_k$$

$\Rightarrow$ local stability of fixed point $\underline{x}^*$ of map $\underline{x}_{k+1} = P(\underline{x}_k)$ guaranteed $\Leftrightarrow |\lambda_j| < 1 \quad \forall\, j \in [1, n-1]$ where λ_j are eigenvalues of $(n-1)\times(n-1)$ Jacobian matrix DP of P

- For the $n \geq 2$ case, fixed points of maps can lose stability in a way impossible in $n=1$.

When $n=1$, $|\lambda| = |f'(x^*)| = 1$ <u>either</u> when $\lambda = 1$ (transcritical bifurcation) <u>or</u> $\lambda = -1$ (flip bifurcation).

When $n \geq 2$, eigenvalues can be complex & when a complex conjugate pair cross the unit circle $|\lambda| = 1$, we have a Neimark-Sacker bifurcation.



- So far we have met <u>smooth</u> maps
  [<u>Smooth</u> means infinitely differentiable, so the map has derivatives of all orders].

$$f(x_n) = x_n e^{-2\pi/\omega} + c_3(1-e^{-2\pi/\omega})$$

linear, not interesting

$$\left.\begin{aligned} f(x_n) &= x_n^2 \\ &= \cos x_n \end{aligned}\right\} \quad \text{nonlinear but not interesting}$$

$$f(x_n) = r x_n (1-x_n)$$

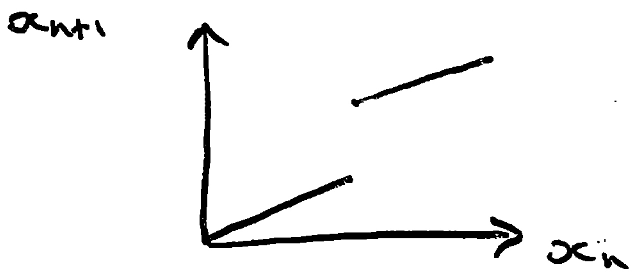
nonlinear and very interesting

- In contrast, non-smooth maps are <u>always</u> interesting
  (nonsmooth, piecewise smooth)

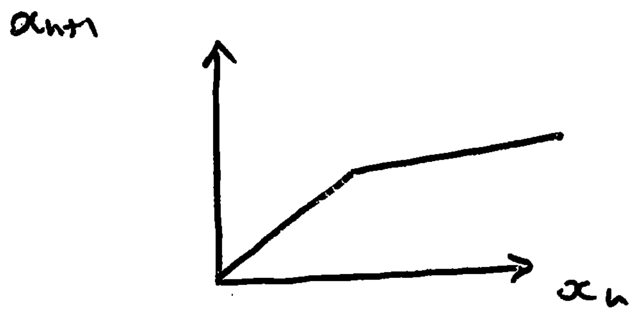
- Nonsmooth maps

    Impacts
    Friction
    Diodes
    Neurons
    Control

- Examples



position discontinuous

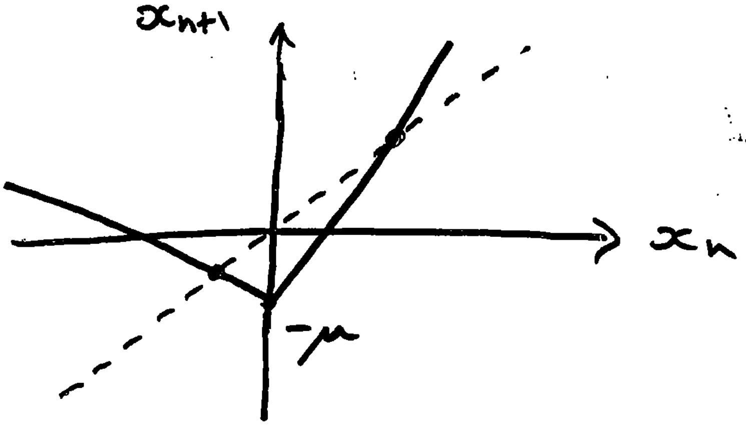


derivative discontinuous

derivative <u>very</u> discontinuous

- Simple case (piecewise linear – PWL)

$$x_{n+1} = \begin{cases} \alpha x_n - \mu & ; \quad x_n \geq 0 \\ \beta x_n - \mu & ; \quad x_n \leq 0 \end{cases}$$

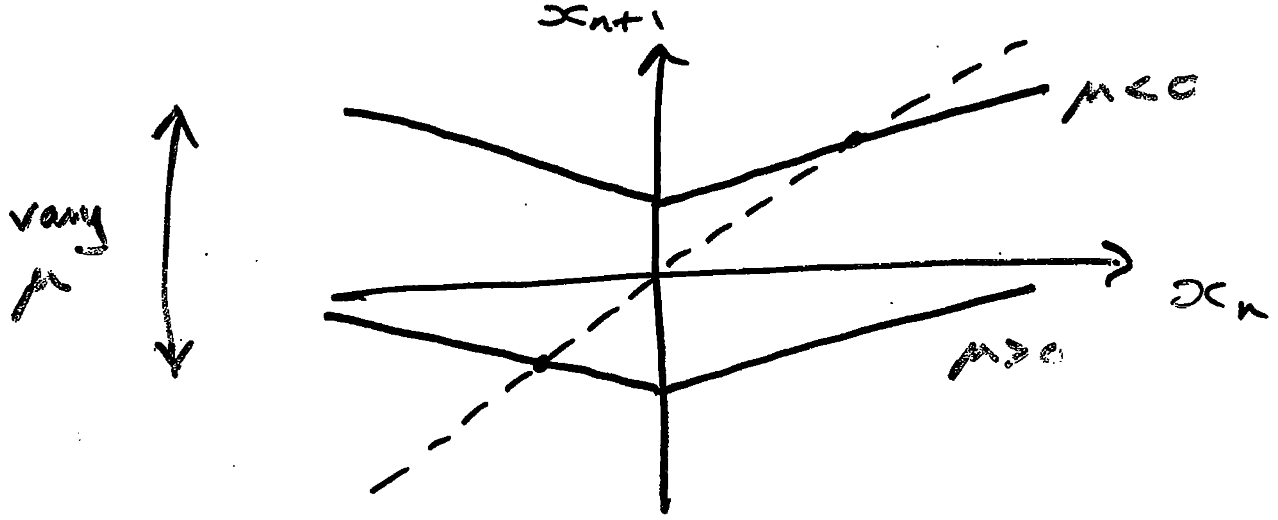


For fixed α, β what happens as the bifurcation parameter μ is varied?

- Since linear, might expect that very little happens. In fact, get more behavior here than in most nonlinear maps!

- In what follows, we take $\alpha > 0$, $\beta < 0$.
[This is the general case, why?]

• Notation:



If we have solution here ($x_n<0$) write

B / b

stable, unstable

If we have solution here ($x_n>0$) write

A/a

stable, unstable

So "B" means "∃ <u>stable</u> solution for $x_n<0$"

etc.

- $\underline{x_n \geqslant 0}$

$$x_{n+1} = \alpha x_n - \mu \qquad (\alpha > 0)$$

Fixed point $x^* = \dfrac{\mu}{\alpha - 1} \geqslant 0$

$x^* \geqslant 0$ exists for $\mu \geqslant 0$
provided $\alpha > 1$ – but is then <u>unstable</u> (why?)

$x^* \geqslant 0$ exists for $\mu \leqslant 0$
provided $0 < \alpha < 1$, and this is <u>stable</u> (why?)

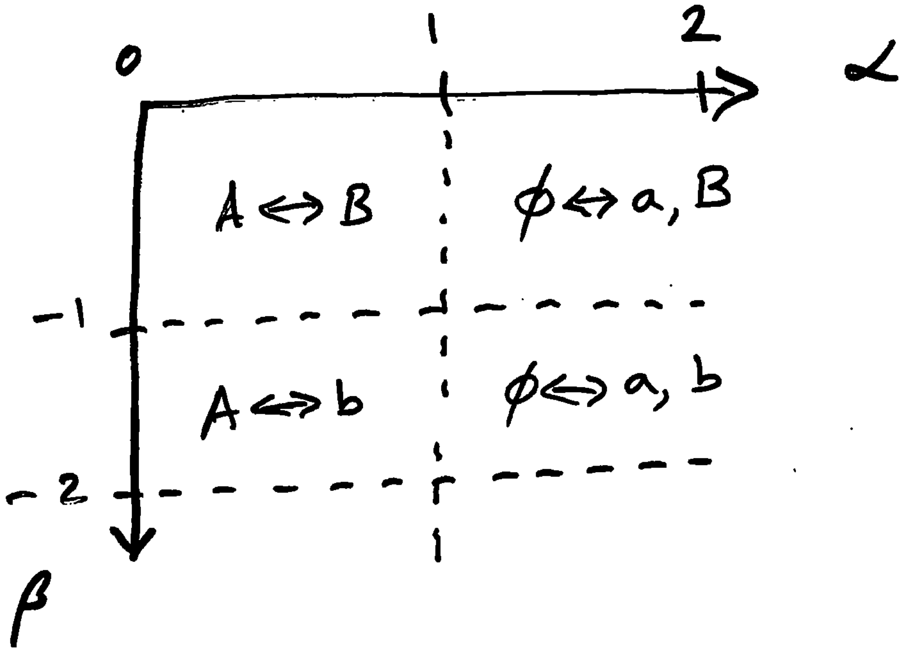
- $\underline{x_n \leqslant 0}$

$$x_{n+1} = \beta x_n - \mu \qquad (\beta < 0)$$

Fixed point $x^{**} = \dfrac{\mu}{\beta - 1} \leqslant 0$

$x^{**} \leqslant 0$ exists provided $\mu \geqslant 0$
(since $\beta < 0$) so if
$\beta > -1 \Rightarrow x^{**}$ stable
$\beta < -1 \Rightarrow x^{**}$ unstable

• Summary



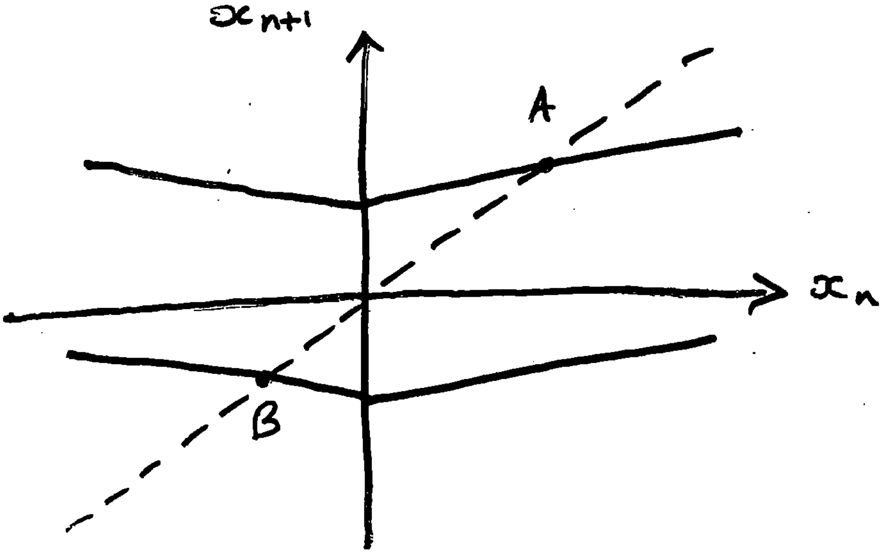
ϕ is empty set, $\Leftrightarrow$ means 'as μ varies from positive to negative or viceversa'.

Example '$A \Leftrightarrow B$' means that for $\alpha < 1$, $\beta > -1$, as μ varies from positive to negative or vice versa, a stable solution with $x > 0$ (A) becomes a stable solution with $x < 0$ (B)

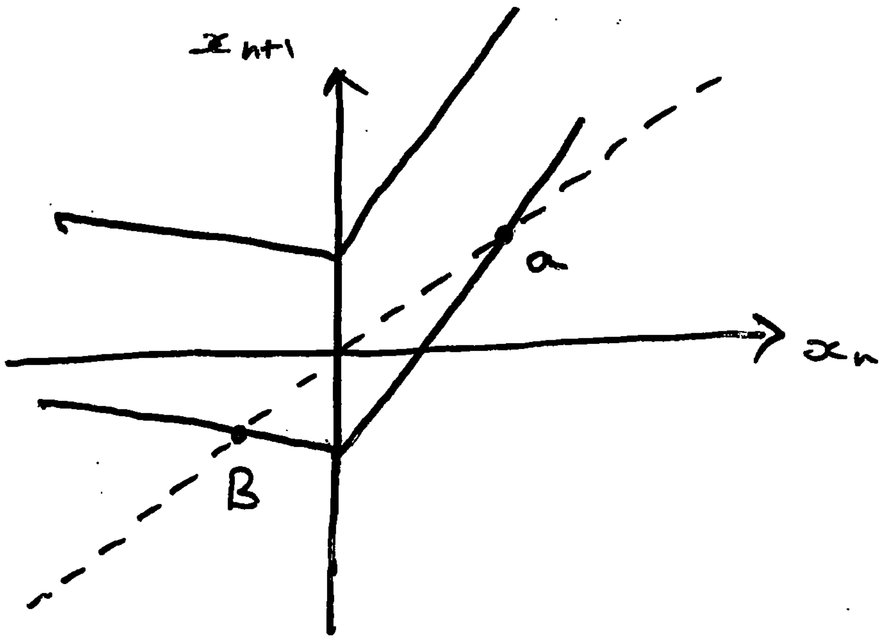
$A \Leftrightarrow B$ and $\phi \Leftrightarrow a, B$
$A \Leftrightarrow b$ $\quad\quad\quad\;$ $\phi \Leftrightarrow a, b$

are our first examples of <u>nonsmooth bifurcations</u>

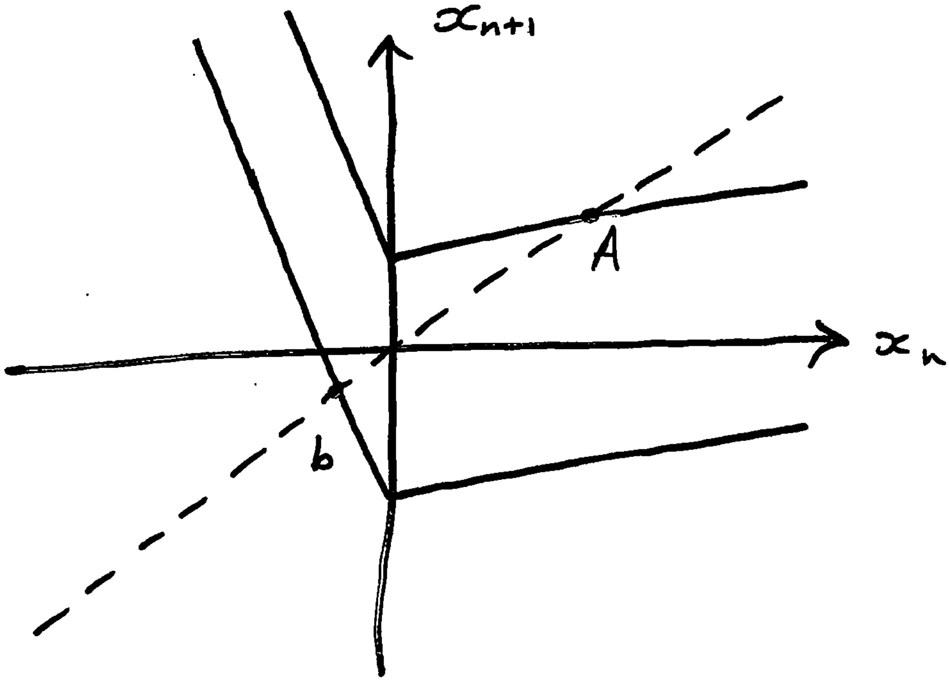
- Another summary:



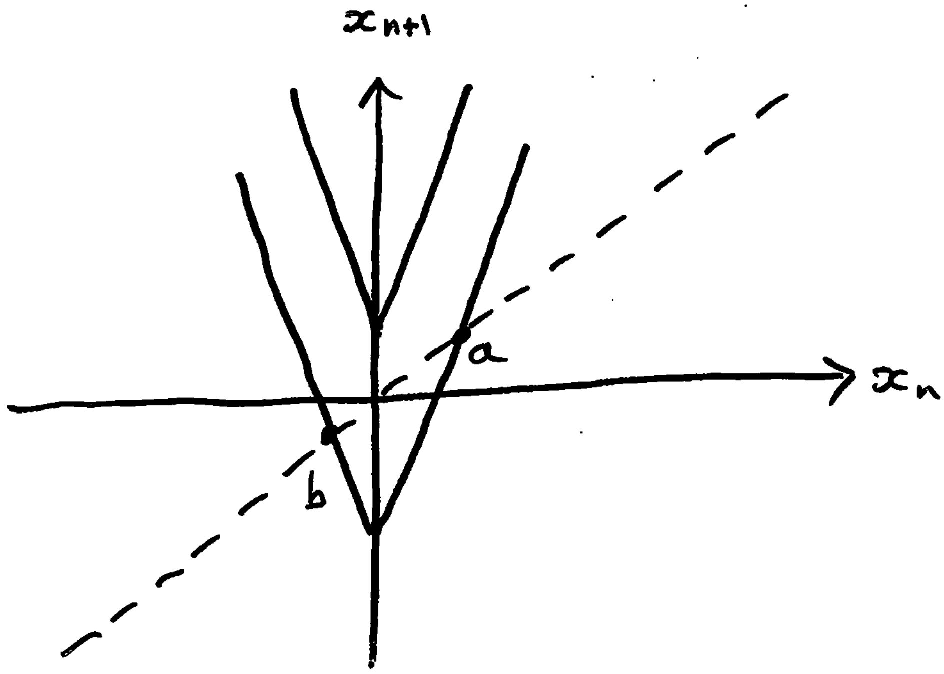
$\alpha < 1$, $\beta > -1$; $A \leftrightarrow B$ smooth transition



$\alpha > 1$, $\beta > -1$; $\phi \leftrightarrow a, B$ annihilation/creation



$\alpha < 1, \beta < -1$; $A \leftrightarrow b$



$\alpha > 1, \beta < -1$; $\phi \leftrightarrow a, b$

- Period doubling can also occur

A linear map is either stable or unstable; period doubling can not occur. So in this case, a period doubling (period 2) solution has to have $x^* < 0$, $x^{**} > 0$ with x^{**} mapped to x^*, x^* mapped back to x^{**}.

$$x^* = \alpha x^{**} - \mu \qquad (x^{**} > 0)$$

$$x^{**} = \beta x^* - \mu \qquad (x^* < 0)$$

Solve to find

$$x^* = \frac{\mu(\alpha+1)}{(\alpha\beta - 1)} < 0$$

$$x^{**} = \frac{\mu(\beta+1)}{(\alpha\beta - 1)} > 0$$

Note this is an solution which is denoted as AB (if stable) or ab (if unstable).

- In terms of the map we have:

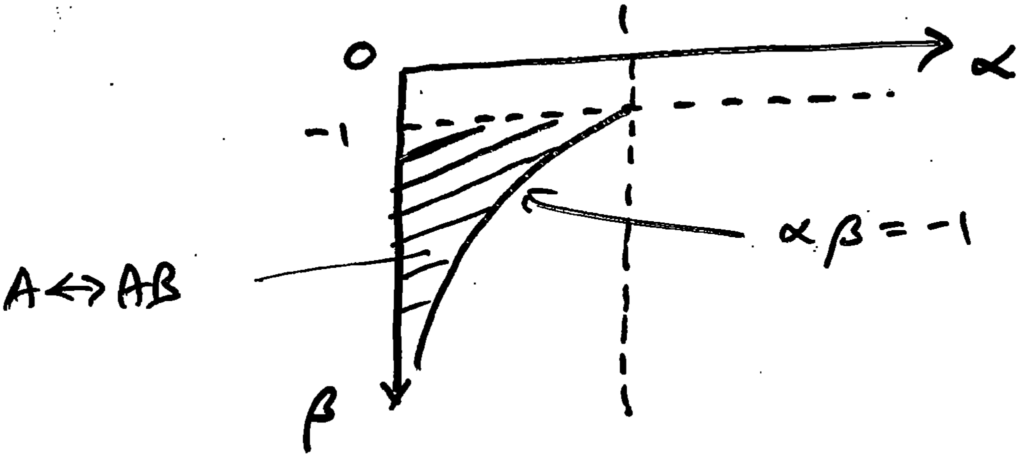
$$x_{n+1} = \alpha x_n - \mu$$

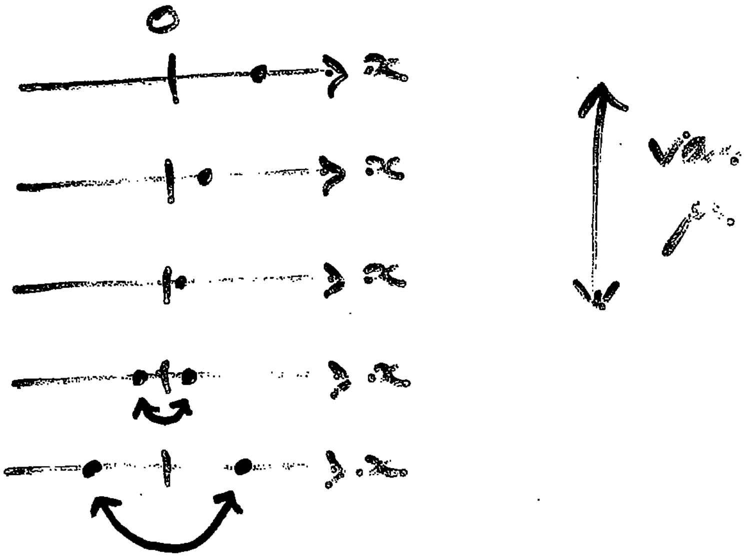
$$x_{n+2} = \beta x_{n+1} - \mu$$

$$\Rightarrow \boxed{x_{n+2} = \alpha\beta x_n - \mu(\beta+1)}$$

So Jacobian (eigenvalue) is $\alpha\beta$

$\Rightarrow (x^*, x^{**})$ is stable when $\alpha\beta > -1$





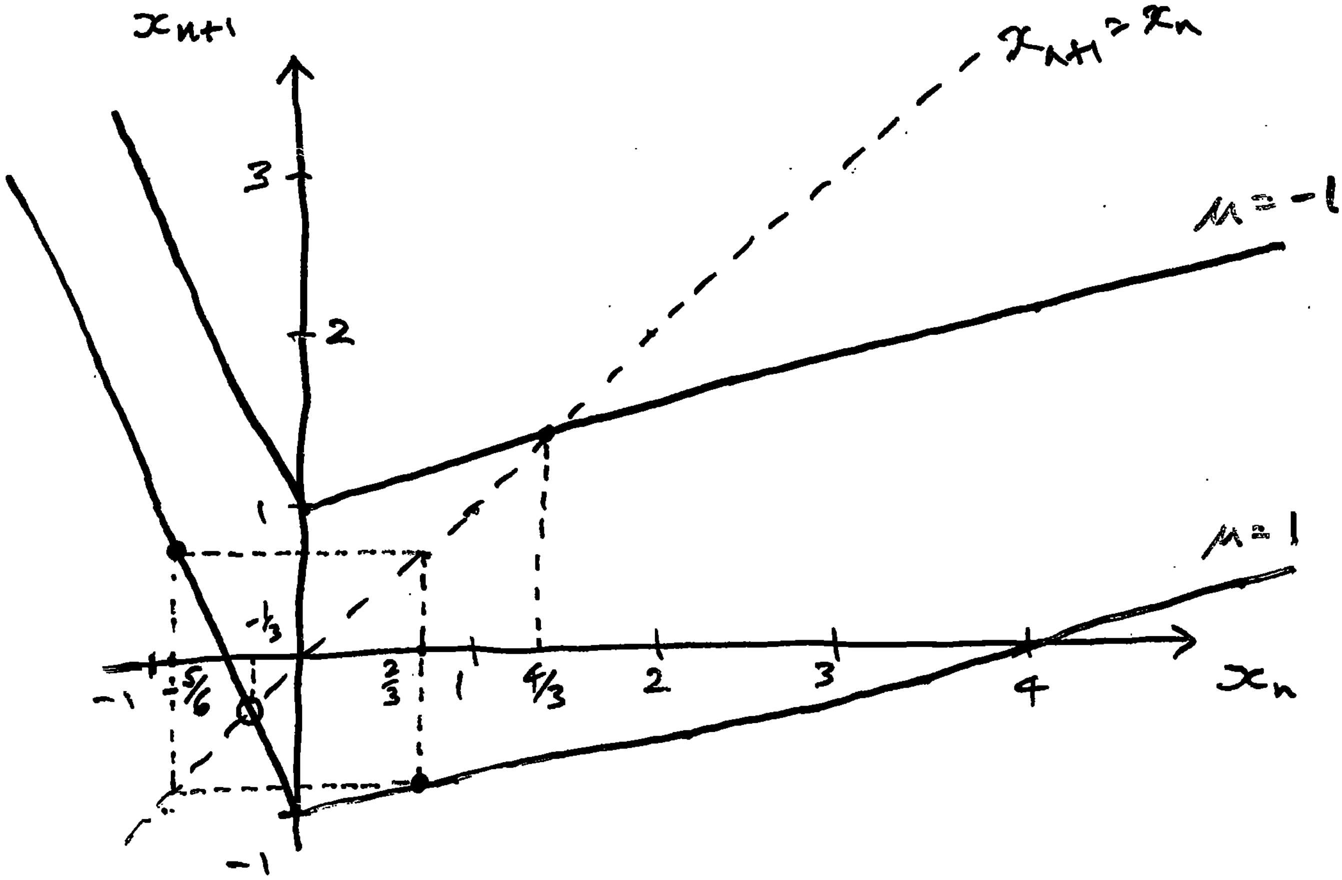
- Example of period doubling:

$$x_{n+1} = \begin{cases} \alpha x_n - \mu & ; x_n \geq 0 \\ \beta x_n - \mu & ; x_n \leq 0 \end{cases} \qquad \alpha = \frac{1}{4}, \beta = -2$$

$$x^* = \frac{\mu(\alpha+1)}{(\alpha\beta - 1)}, \qquad x^{**} = \frac{\mu(\beta+1)}{(\alpha\beta-1)}$$

$$\Rightarrow x^* = -\frac{5}{6}, \quad x^{**} = \frac{2}{3} \quad \text{when } \mu = 1$$

can show (see later) that this solution does not exist for $\mu < 0$ where there is only one fixed point ($x^* = \frac{4}{3}$ when $\mu = -1$)



This is $A \Leftrightarrow b, AB$

$\frac{4}{3} \Leftrightarrow -\frac{1}{3}, \left\{\frac{2}{3}, -\frac{5}{6}\right\}$

· Example of period adding $(1 \to 3)$

$$x_{n+1} = \begin{cases} \alpha x_n - \mu \; ; & x_n \geq 0 \\ \beta x_n - \mu \; ; & x_n \leq 0 \end{cases}$$

Illustrate $A \to A^2 B$

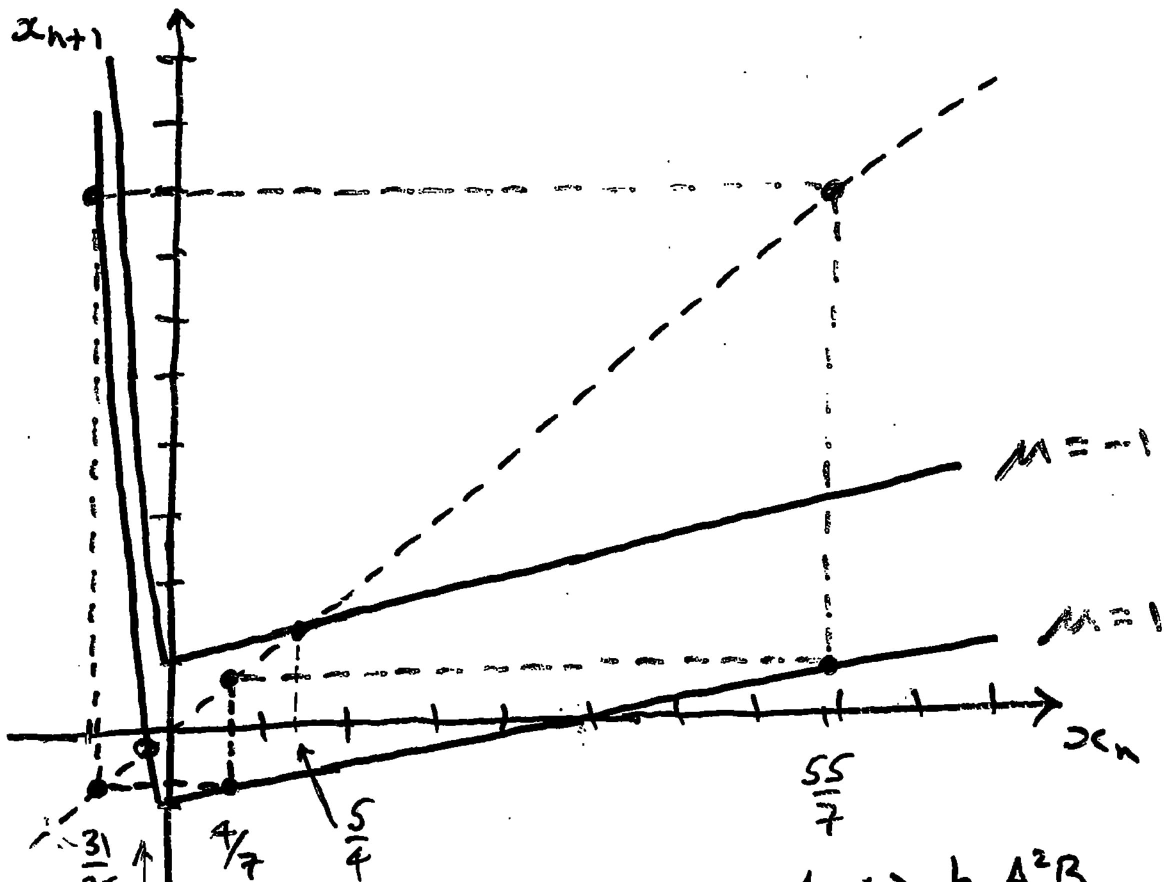
$$x_2 = \beta x_1 - \mu \; ; \quad x_1 < 0$$
$$x_3 = \alpha x_2 - \mu \; ; \quad x_2 > 0$$
$$x_1 = \alpha x_3 - \mu \; ; \quad x_3 > 0$$

$$\Rightarrow x_1 = \frac{\mu(1+\alpha+\alpha^2)}{(\alpha^2\beta - 1)}, \quad x_2 = \frac{\mu(1+\beta+\alpha\beta)}{(\alpha^2\beta-1)}, \quad x_3 = \frac{\mu(1+\alpha+\alpha\beta)}{(\alpha^2\beta-1)}$$

Need $\mu > 0$. $\alpha = 1/5$, $\beta = -10$, $\mu = 1$

$\Rightarrow x_1 = -31/35$, $x_2 = 55/7$, $x_3 = 4/7$

when $\mu = -1$, $x = 5/4$ only.



$-\frac{1}{11}$

This is $A \leftrightarrow b, A^2 B$

$$\frac{5}{4} \leftrightarrow -\frac{1}{11}, \left\{\frac{55}{7}, \frac{4}{7}, -\frac{31}{35}\right\}$$

- Nonsmooth bifurcations in n-dimensional piecewise linear maps.

We are going to obtain general conditions for nonsmooth bifurcations. Let $\underline{x} = (x_1, x_2, \cdots, x_n)^T$ be an n-dimensional vector. Let $\underline{x}^{(k)}$ be the k^{th} iterate of this vector.

The map is

$$\underline{x}^{(k+1)} = \begin{cases} \underline{\underline{A}}_1 \underline{x}^{(k)} + \underline{c}\mu & ; \quad x_n^{(k)} \geq 0 \\ \underline{\underline{A}}_2 \underline{x}^{(k)} + \underline{c}\mu & ; \quad x_n^{(k)} \leq 0 \end{cases}$$

where

$\underline{\underline{A}}_1, \underline{\underline{A}}_2$ are (real) $n \times n$ matrices

$\underline{c}$ is a (real) $n \times 1$ vector

μ is a (real) parameter

Note: mapping is continuous when $x_n^{(k)} = 0 \quad \forall k$

and smooth for $x_n^{(k)} = 0$, $\mu = 0$; $k = 1, \cdots, n-1$

$\Rightarrow \underline{\underline{A}}_1 \equiv [a_{ij}^{(1)}]$, $\underline{\underline{A}}_2 \equiv [a_{ij}^{(2)}]$ are identical except for the last column $\Rightarrow a_{ij}^{(1)} = a_{ij}^{(2)}$; $j \neq n$

Let M^*, M^{**} be fixed points of the map such that

$$M^* = A_1 M^* + c\mu \quad ; \quad m_n^* > 0$$

$$M^{**} = A_2 M^{**} + c\mu \quad ; \quad m_n^{**} < 0$$

where $m_k^* = [M^*]_k$, $m_k^{**} = [M^{**}]_k$.

Assuming $(A_1 - I), (A_2 - I)$ to be invertible $\Rightarrow$

$$M^* = -\frac{\mathrm{adj}(A_1 - I)}{p^*(1)} c\mu, \quad M^{**} = \frac{-\mathrm{adj}(A_2 - I)}{p^{**}(1)} c\mu$$

where $p^*(1)$ is characteristic polynomial of A_1, evaluated at 1, $p^{**}(1)$ is characteristic polynomial of A_2, evaluated at 1.†

In scalar form, this becomes

$$m_k^* = \frac{b_k^*}{p^*(1)} \mu, \quad m_k^{**} = \frac{b_k^{**}}{p^{**}(1)} \mu$$

where

$$b_k^* = [-\mathrm{adj}(A_1 - I) c]_k$$

$$b_k^{**} = [-\mathrm{adj}(A_2 - I) c]_k$$

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† Alternatively $p^*(\lambda) = \det(A_1 - \lambda I)$
$p^{**}(\lambda) = \det(A_2 - \lambda I)$