

HILBERT FUNCTIONS OF IRREDUCIBLE ARITHMETICALLY GORENSTEIN SCHEMES

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ABSTRACT. In this paper we compute the Hilbert functions of irreducible (or smooth) and reduced arithmetically Gorenstein schemes that are twisted anti-canonical divisors on arithmetically Cohen-Macaulay schemes. We also prove some folklore results characterizing the Hilbert functions of irreducible standard determinantal schemes, and we use them to produce a new class of functions that occur as Hilbert functions of irreducible (or smooth) and reduced arithmetically Gorenstein schemes in any codimension.

INTRODUCTION

There is a simple characterization of the functions that arise as Hilbert functions of arithmetically Cohen-Macaulay schemes. Nevertheless, very little is known about the Hilbert functions of *irreducible* arithmetically Cohen-Macaulay schemes. Harris proved that the h -vector of an irreducible aCM scheme of positive dimension is the same as the h -vector of a zero-scheme satisfying the Uniform Position Property (see [13]). However, no characterization is available yet for these h -vectors. In codimension 2, the question has been completely answered following a different approach. Notice that in codimension 2, due to the Hilbert-Burch theorem, arithmetically Cohen-Macaulay schemes and standard determinantal schemes coincide. A standard determinantal scheme is defined by the (homogeneous) maximal minors of a suitable homogeneous matrix of forms, see Definition 2.1. In the first part of the paper, we state some folklore facts about irreducible standard determinantal schemes, leaving the proofs for the Appendix. First, we express the Hilbert function of any standard determinantal scheme in terms of its degree matrix (Proposition 2.4). Then, we characterize the Hilbert functions of irreducible and reduced standard determinantal schemes in \mathbf{P}^n of any codimension in terms of the entries of the degree matrix (Theorem 2.8). In particular, since any standard determinantal scheme is an arithmetically Cohen-Macaulay scheme, we obtain a large class of numerical functions that occur as the Hilbert functions of some irreducible and reduced arithmetically Cohen-Macaulay schemes.

If one restricts attention to *irreducible* arithmetically Gorenstein schemes, the question of characterizing their Hilbert functions has been answered only in the codimension

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3 case, thanks to the Buchsbaum-Eisenbud structure theorem (see [7]). However, the question is still open in higher codimension. In the second part of the paper, we produce a new class of Hilbert functions that occur for irreducible and reduced (respectively, irreducible and smooth) arithmetically Gorenstein schemes of any codimension (Corollary 3.4, and Corollary 3.5). The strategy is to show (Theorem 3.2) that, for an aCM subscheme $S \subset \mathbf{P}^n$ which is Gorenstein in codimension one, a general element of the linear system $|mH - K|$ determines an irreducible arithmetically Gorenstein divisor whose h -vector can be written in terms of the Hilbert function of S . Here H is a hyperplane section of S by a hyperplane that meets it properly, K a canonical divisor, and $m \gg 0$ (Theorem 3.2 also contains an estimate of how big m can be chosen). The Corollaries mentioned above are obtained by combining this result with the folklore results of the first part.

In the first section we recall a few facts about Hilbert functions. In the second section we state the folklore facts mentioned above. In the third section we draw our main conclusions. The Appendix contains the proofs omitted in Section 2.

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1. PRELIMINARIES

Let S be a closed subscheme of the projective n -space $\mathbf{P}^n = \mathbf{P}^n(k)$, where k is an algebraically closed field. Let I_S be the saturated homogeneous ideal corresponding to S in the polynomial ring $R = k[x_0, \dots, x_n]$.

The numerical function

$$\begin{aligned} H_S : \mathbf{N} &\longrightarrow \mathbf{N} \\ n &\longmapsto \dim_k(R/I_S)_n \end{aligned}$$

is called the *Hilbert function* of S . The formal series

$$P_S(z) = \sum_{n \geq 0} H_S(n)z^n$$

is called the *Hilbert series* of S . It is well-known that the Hilbert series of S can be expressed in the rational form

$$P_S(z) = \frac{h_S(z)}{(1-z)^{d+1}},$$

where $h_S(z)$ is a polynomial with integer coefficients such that $h_S(1) = \deg S$ is the degree of the scheme, and d is the dimension of S . The polynomial

$$h_S(z) = \sum_{i=0}^s h_i z^i,$$

with $h_s \neq 0$, is called the *h -polynomial* of S , and the vector (h_0, \dots, h_s) defined by the coefficients of $h_S(z)$ is called the *h -vector* of S . We will say that an h -vector $h = (h_0, \dots, h_s)$ has length s .

By an *arithmetically Cohen-Macaulay* (abbreviated aCM) projective scheme we mean a projective scheme whose coordinate ring is Cohen-Macaulay. For an aCM scheme S , set $H_S(-1) = 0$, and define

$$\begin{aligned}\Delta^1 H_S(t) &= H_S(t) - H_S(t-1), \\ \Delta^r H_S(t) &= \Delta^{r-1} H_S(t) - \Delta^{r-1} H_S(t-1).\end{aligned}$$

$\Delta^r H_S$ is called the r -th *difference* of $H_S(t)$ and it is the Hilbert function of the r -th general hyperplane section X of S .

The problem of characterizing those numerical functions that occur as Hilbert functions of schemes with given properties has been studied extensively. There is a simple characterization of the Hilbert series of aCM projective schemes (see e.g. [25], Theorem 1.5). However, in general very little is known about the Hilbert series of irreducible aCM schemes. The case of irreducible aCM schemes of codimension 2 is better understood, thanks to the structure theorem of Hilbert and Burch (see [5], Theorem 1.4.17). The numerical functions that can occur as Hilbert functions for reduced, irreducible aCM schemes of codimension 2 are characterized in [12] and in [25], Theorem 2.3. There is no analogous characterization in higher codimension.

By an *arithmetically Gorenstein* (abbreviated aG) scheme we mean a projective scheme whose coordinate ring is Gorenstein. In particular, any aG scheme is an aCM scheme. Even in the case of arithmetically Gorenstein projective schemes there is no complete characterization of the Hilbert series. A necessary, but not sufficient, condition for a polynomial to be the h -polynomial of some aG scheme is symmetry in the coefficients, i.e. if $h(z) = 1 + h_1 z + \dots + h_{s-1} z^{s-1} + h_s z^s$, then $h_s = 1$ and $h_i = h_{s-i}$ for all $i = 1, \dots, s-1$.

For codimension three aG schemes, the Hilbert series can be characterized using the structure theorem of Buchsbaum-Eisenbud (see [25], Theorem 2.18). For irreducible codimension three aG schemes, a characterization of the Hilbert series is given by De Negri and Valla in [7]. Our Corollary 3.4 gives new examples in each codimension greater than three of Hilbert series occurring for irreducible aG schemes. Corollary 3.5 does the same, under the extra assumption that the irreducible aG schemes be smooth.

Finally, we briefly recall the definition of Castelnuovo-Mumford regularity.

Definition 1.1. A coherent sheaf \mathcal{F} on \mathbf{P}^n is said to be m -regular if

$$H^i(\mathbf{P}^n, \mathcal{F}(m-i)) = 0$$

for all $i > 0$. The *regularity* or *Castelnuovo-Mumford regularity* of \mathcal{F} is

$$\text{reg}(\mathcal{F}) = \min\{m \mid \mathcal{F} \text{ is } m\text{-regular}\}.$$

By a theorem of Serre ([14], Theorem 3.5.2), any \mathcal{F} is m -regular for some m . It follows from the Castelnuovo-Mumford Theorem that if \mathcal{F} is an m -regular coherent sheaf on \mathbf{P}^n , then $\mathcal{F}(k)$ is generated as an $\mathcal{O}_{\mathbf{P}^n}$ -module by its global sections for all $k \geq m$ (see e.g. [21], Theorem 1.1.5).

If $V \subset \mathbf{P}^n$ is a subscheme we define the *Castelnuovo-Mumford regularity* of V as the Castelnuovo-Mumford regularity of its ideal sheaf, $\text{reg}(V) := \text{reg}(\mathcal{I}_V)$.

Remark 1.2. (i) It is well known (and easy to prove) that if

$$0 \longrightarrow \dots \longrightarrow \bigoplus_j R(-a_{i,j}) \longrightarrow \dots \longrightarrow \bigoplus_j R(-a_{0,j}) \rightarrow I_V \longrightarrow 0$$

is a graded minimal free R -resolution of I_V , then

$$\operatorname{reg}(\mathcal{I}_V) = \max_{i,j} \{a_{i,j} - i\}$$

(see for instance [21], Remark 1.1.6).

(ii) The Castelnuovo-Mumford regularity of a scheme $S \subset \mathbf{P}^n$ should not be confused with the *regularity index* of a scheme, $r(S)$, that is the minimum degree in which the Hilbert function of S agrees with the Hilbert polynomial (see [5] for more details). If $S \subset \mathbf{P}^n$ is an aCM scheme of dimension d , then $r(S) = \operatorname{reg}(S) - d - 1$ (this follows from [5], Theorem 4.4.3 (b)).

2. HILBERT FUNCTIONS OF STANDARD DETERMINANTAL SCHEMES

The results stated in this section are regarded as folklore. We will compute the Hilbert function of a standard determinantal scheme in terms of the degree matrix associated to it. In particular, we will derive formulas for the degree and the Castelnuovo-Mumford regularity of the scheme. Complete proofs using standard methods are given in the Appendix.

Definition 2.1. A subscheme $S \subset \mathbf{P}^n$ is called *standard determinantal* if I_S is generated by the maximal minors of an $l \times (l + c - 1)$ homogeneous matrix $M = (g_{ij})$ representing a morphism

$$\phi : F = \bigoplus_{j=1}^{l+c-1} R(a_j) \longrightarrow G = \bigoplus_{j=1}^l R(b_j)$$

of free graded R -modules. Here c is the codimension of S and we assume that $a_1 \leq \dots \leq a_{l+c-1}$ and $b_1 \leq \dots \leq b_l$. The degree matrix of M is the matrix $U = (u_{ij})$ whose entries are the degrees of the entries of M . We will call *degree matrix* any matrix of integers that is the degree matrix associated to some homogeneous matrix of polynomials.

Remark 2.2. (i) In the notation above, the entries of U increase from right to left and from top to bottom: $u_{i,j} \geq u_{k,r}$ if $i \geq k$ and $j \leq r$.

(ii) We will assume without loss of generality that the degree matrix has the property $g_{ij} = 0$ if $u_{ij} \leq 0$, and $\deg g_{ij} = u_{ij}$ if $u_{ij} > 0$. Note that g_{ij} could be 0 even if $u_{ij} > 0$.

(iii) Since $U = (u_{ij})$ is the degree matrix of a homogeneous matrix, one has

$$(1) \quad \sum_{v=1}^s u_{i_v, j_v} = \sum_{v=1}^s u_{i_v, j_{\pi(v)}},$$

for every permutation π of $\{1, \dots, s\}$.

Any standard determinantal scheme is an aCM scheme and its minimal free resolution is given by the Eagon-Northcott complex (see proof of Proposition 2.4 in the Appendix). Moreover, in codimension 2, any aCM scheme is a standard determinantal scheme due to the Hilbert-Burch Theorem.

Next, we show how to recover the Hilbert function of a standard determinantal scheme from its degree matrix.

Notation 2.3. Let $U = (u_{ij})$ be a matrix of size $l \times (l + c - 1)$. By an $m \times m$ submatrix V of U we mean the $m \times m$ matrix obtained from the elements of U given a choice of m rows and m columns. For a square matrix V we say that $\dim V = m$ if V is a $m \times m$ matrix. By $(V_1|V_2|\dots) \subset U$ we mean a choice of square submatrices V_j of U with $\dim V_1 = l$, such that: the choice of columns for V_{j+1} is strictly to the right-hand side of any column chosen for V_j , and the choice of rows for V_{j+1} is made from the choice of rows for V_j (so $0 \leq \dim V_{j+1} \leq \dim V_j \leq l$). Denote by $\text{tr}(V_1|V_2|\dots)$ the sum of the traces of V_j (see Example 2.6).

With the notation above, we have the following result whose proof is left to the Appendix. We thank the referee who pointed out a similar formula obtained by Ausina and Ballesteros in the unpublished paper [2] (see also Section 5 of [3]). Our results were obtained independently, and we give different formulas than those of Ausina and Ballesteros. The main tool used here is the Eagon-Northcott resolution, as in [2].

Proposition 2.4. *Let $S \subset \mathbf{P}^n$ be a standard determinantal scheme of codimension c with degree matrix $U = (u_{i,j})$, $i = 1, \dots, l$, $j = 1, \dots, l + c - 1$. Then*

$$H_S(t) = \binom{t+n}{n} + \sum_{(V_1|V_2|\dots) \subset U} (-1)^{1+\dim V_2+\dots} \binom{t+n - \text{tr}(V_1|V_2|\dots)}{n},$$

$$\deg(S) = 1 + \frac{1}{c!} \sum_{(V_1|V_2|\dots) \subset U} (-1)^{c+1+\dim V_2+\dots} (\text{tr}(V_1|V_2|\dots) - 1) \cdot \dots \cdot (\text{tr}(V_1|V_2|\dots) - c).$$

The Castelnuovo-Mumford regularity of S is

$$\text{reg}(S) = \text{tr}(V_1^0|V_2^0|\dots|V_c^0) - c + 1,$$

where V_1^0 is the submatrix of U formed by the first l columns, and $V_{k+1}^0 = (u_{l,l+k})$ for $k = 1, \dots, c - 1$ (i.e. V_{k+1}^0 is the $(l+k)$ -th element in the last row of U).

Remark 2.5. (i) The expression for $\deg S$ in Proposition 2.4 is a new version of Porteus' formula in the case of standard determinantal schemes (see [1], II.4.2). The advantage is that this formula involves only the entries of the degree matrix of S , while Porteus' formula involves Chern classes.

(ii) From Proposition 2.4 and Remark 1.2 we have that the index of regularity of S is

$$r(S) = \text{tr}(V_1^0|V_2^0|\dots|V_c^0) - n.$$

Example 2.6. Let $S \subset \mathbf{P}^5$ be a standard determinantal scheme of codimension 3 with degree matrix

$$U = \begin{pmatrix} 2 & 2 & 2 & 1 \\ 3 & 3 & 3 & 2 \end{pmatrix}$$

To compute the degree and the regularity of S we have to consider the following combinations of submatrices of U :

$$\left(\begin{array}{cc|cc} 2 & 2 & \cdot & \cdot \\ 3 & 3 & \cdot & \cdot \end{array} \right), \left(\begin{array}{cc|c|c} 2 & 2 & 2 & \cdot \\ 3 & 3 & \cdot & \cdot \end{array} \right), \left(\begin{array}{cc|c|c} 2 & 2 & 2 & 1 \\ 3 & 3 & \cdot & \cdot \end{array} \right), \left(\begin{array}{cc|c|c} 2 & 2 & \cdot & \cdot \\ 3 & 3 & 3 & \cdot \end{array} \right), \left(\begin{array}{cc|c|c} 2 & 2 & \cdot & \cdot \\ 3 & 3 & 3 & 2 \end{array} \right),$$

$$\begin{aligned} & \left(\begin{array}{cc|cc} 2 & 2 & 2 & 1 \\ 3 & 3 & 3 & 2 \end{array} \right), \left(\begin{array}{cc|cc} 2 & 2 & \cdot & 1 \\ 3 & 3 & \cdot & \cdot \end{array} \right), \left(\begin{array}{cc|cc} 2 & 2 & \cdot & \cdot \\ 3 & 3 & \cdot & 2 \end{array} \right), \left(\begin{array}{ccc|c} 2 & \cdot & 2 & \cdot \\ 3 & \cdot & 3 & \cdot \end{array} \right), \left(\begin{array}{ccc|c} 2 & \cdot & 2 & 1 \\ 3 & \cdot & 3 & \cdot \end{array} \right), \\ & \left(\begin{array}{ccc|c} 2 & \cdot & 2 & \cdot \\ 3 & \cdot & 3 & 2 \end{array} \right), \left(\begin{array}{ccc|c} 2 & \cdot & \cdot & 1 \\ 3 & \cdot & \cdot & 2 \end{array} \right), \left(\begin{array}{ccc|c} \cdot & 2 & 2 & \cdot \\ \cdot & 3 & 3 & \cdot \end{array} \right), \left(\begin{array}{ccc|c} \cdot & 2 & 2 & 1 \\ \cdot & 3 & 3 & \cdot \end{array} \right), \left(\begin{array}{ccc|c} \cdot & 2 & 2 & \cdot \\ \cdot & 3 & 3 & 2 \end{array} \right), \\ & \left(\begin{array}{ccc|c} \cdot & 2 & \cdot & 1 \\ \cdot & 3 & \cdot & 2 \end{array} \right), \left(\begin{array}{ccc|c} \cdot & \cdot & 2 & 1 \\ \cdot & \cdot & 3 & 2 \end{array} \right). \end{aligned}$$

The degree of S is

$$\begin{aligned} \deg(S) &= 1 + \frac{1}{6} (4 \cdot 3 \cdot 2 - 6 \cdot 5 \cdot 4 + 7 \cdot 6 \cdot 5 - 7 \cdot 6 \cdot 5 + 9 \cdot 8 \cdot 7 + 8 \cdot 7 \cdot 6 - 5 \cdot 4 \cdot 3 - 6 \cdot 5 \cdot 4 \\ &+ 4 \cdot 3 \cdot 2 - 5 \cdot 4 \cdot 3 - 6 \cdot 5 \cdot 4 + 3 \cdot 2 \cdot 1 + 4 \cdot 3 \cdot 2 - 5 \cdot 4 \cdot 3 - 6 \cdot 5 \cdot 4 + 3 \cdot 2 \cdot 1 + 3 \cdot 2 \cdot 1) = 46. \end{aligned}$$

For the degree matrix U we have

$$(V_1^0 | V_2^0 | V_3^0) = \left(\begin{array}{cc|cc} 2 & 2 & \cdot & \cdot \\ 3 & 3 & 3 & 2 \end{array} \right),$$

hence Proposition 2.4 tells us that the Castelnuovo-Mumford regularity of S is $\text{reg}(S) = 10 - 3 + 1 = 8$.

We focus now on determining the numerical functions that are Hilbert functions of standard determinantal schemes with interesting properties such as reduced and irreducible.

Next, given a matrix satisfying certain positivity conditions on the entries, we construct a reduced standard determinantal scheme that has this degree matrix. The existence of such a scheme also follows from work of Trung ([24]). A self-contained proof of the following proposition is given in the Appendix.

Proposition 2.7. *Let $U = (u_{i,j})$ be a degree matrix of size $l \times (l + c - 1)$, satisfying the condition (ii) of Remark 2.2. Suppose that $u_{i,i+c-1} > 0$, $i = 1, \dots, l$. Then, for any n with $n \geq c \geq 1$, there exists a reduced standard determinantal scheme $X \subset \mathbf{P}^n$ of codimension c with degree matrix U .*

We can now obtain a large class of Hilbert functions that occur as Hilbert functions of reduced and irreducible, arithmetically Cohen-Macaulay schemes.

Theorem 2.8. *Let $U = (u_{i,j})$ be a degree matrix of size $l \times (l + c - 1)$. Let n be an integer, $n > c \geq 1$. Then*

$$H(t) = \binom{t+n}{n} + \sum_{(V_1|V_2|\dots) \subset U} (-1)^{1+\dim V_2+\dots} \binom{t+n - \text{tr}(V_1|V_2|\dots)}{n}$$

is the Hilbert function of a non-degenerate, irreducible and reduced standard determinantal scheme $X \subset \mathbf{P}^n$ if and only if $u_{i,i+c} > 0$ for $i = 1, \dots, l - 1$.

Proof. It follows from a theorem of Trung (see [24], Theorem 4.4) that if $S \subset \mathbf{P}^n$ is a standard determinantal scheme of codimension $c \geq 2$ with degree matrix U , then S is the hyperplane section of a normal, standard determinantal reduced, irreducible scheme $S' \subset \mathbf{P}^{n+1}$ of codimension c by a hyperplane that meets it properly if and only if $u_{i,i+c} > 0$ for $i = 1, \dots, l-1$. Clearly, S and S' will have the same degree matrix. Then the theorem follows directly from Proposition 2.4. \square

3. HILBERT FUNCTIONS OF IRREDUCIBLE ARITHMETICALLY GORENSTEIN SCHEMES

In this section we obtain a set of functions that are Hilbert functions of a large class of irreducible and reduced arithmetically Gorenstein schemes. In order to be able to use Bertini's Theorem (as in Theorem 3.2), we will be working over an algebraically closed field k of characteristic 0.

Recall that a noetherian ring A (respectively, a noetherian scheme X) satisfies the condition G_r , *Gorenstein in codimension less or equal r* , if every localization A_P at a prime ideal $P \subset A$ (respectively, every local ring $\mathcal{O}_{X,x}$) of codimension less than or equal to r is a Gorenstein local ring. In other words, the non locally-Gorenstein locus has codimension greater than r (see [15] for more details).

Let $S \subset \mathbf{P}^n$ be a codimension c scheme satisfying property G_1 , and let I_S be its saturated homogeneous ideal. A *divisor* D on S is a generalized divisor in the sense of [15].

We will denote a dualizing sheaf on S by ω_S and the corresponding canonical divisor by K . We will denote the canonical module of S by K_S , that is

$$K_S = \text{Ext}_R^c(R/I_S, R)(-n-1)$$

which is isomorphic to $H_*^0(\omega_S)$. Here, $H_*^i(\mathcal{F}) := \bigoplus_{t \in \mathbf{Z}} H^i(\mathbf{P}^n, \mathcal{F}(t))$ for any sheaf \mathcal{F} . Finally, we will denote by ω_S^\vee the \mathcal{O}_S -dual of the canonical sheaf ω_S .

In order to construct arithmetically Gorenstein schemes we will use the following result:

Proposition 3.1. *(Corollary 5.5, [18]) Let $S \subset \mathbf{P}^n$ be an aCM subscheme satisfying G_1 , K a canonical divisor on S , and H the hyperplane section. Then every element of the linear system $|mH - K|$ is arithmetically Gorenstein.*

Next, we compute the Hilbert functions of these arithmetically Gorenstein schemes, in order to obtain numerical functions that occur as Hilbert functions of irreducible arithmetically Gorenstein schemes. We express the Hilbert functions of these aG divisors on a scheme S only in terms of the Hilbert function of S and its regularity. Notice that all the h -vectors that arise this way are of *decreasing type*, in the sense that if $\Delta h_Y(j_0) < 0$ for some j_0 then $\Delta h_Y(j) < 0$ for all $j \geq j_0$.

While it has been proved that the h -vector of an arithmetically Gorenstein, reduced and irreducible scheme of codimension 3 is of decreasing type (see [7]), it is not known whether the same holds in codimension 4 or higher.

Theorem 3.2. *Let $S \subset \mathbf{P}^n$ be an irreducible and reduced aCM scheme of dimension $d \geq 2$, satisfying property G_1 . Denote by $\Delta^d H_S$ the Hilbert function of the d^{th} -general hyperplane section of S . Set*

$$r := \min \{i | \Delta^d H_S(i) = \deg S\} = \text{reg}(S) - 1.$$

Let Y be a general element in the linear system $|mH - K|$ for $m \geq \{2r - d, \text{reg}(\omega_S^\vee)\}$. Then Y is an irreducible and reduced arithmetically Gorenstein scheme whose h -vector is

of decreasing type and satisfies

$$h_Y(t) = \begin{cases} \Delta^d H_S(t), & t \leq r \\ \deg S, & r \leq t \leq m - r + d \\ \Delta^d H_S(m - t + d), & t \geq m - r + d. \end{cases}$$

Proof. Let $Y \subset S$ be a general element of the linear system $|mH - K|$, $m \in \mathbf{Z}$. For $m \geq \text{reg}(\omega_S^\vee)$, this linear system is base point free and, by Bertini's Theorem (see [17], p. 89), the general element Y is irreducible. Moreover, Y is arithmetically Gorenstein by Proposition 3.1. Let I_S and I_Y be the saturated homogeneous ideals of S and Y as subschemes of \mathbf{P}^n and let $\mathcal{I}_{Y,S}$ be the sheafification of the ideal $I_Y/I_S \subset R/I_S$. We have

$$\mathcal{I}_{Y,S} \cong \mathcal{O}_S(K - mH) \cong \omega_S(-m)$$

and S is aCM of dimension > 1 . Therefore $H_*^1(\mathcal{I}_S) = 0$, and we obtain the exact sequence

$$0 \rightarrow I_S \rightarrow I_Y \rightarrow H_*^0(\omega_S)(-m) \rightarrow 0$$

by taking cohomology in $0 \rightarrow \mathcal{I}_S \rightarrow \mathcal{I}_Y \rightarrow \mathcal{I}_{Y,S} \rightarrow 0$. Thus we get the following equality on Hilbert functions for every t :

$$(2) \quad H_Y(t) = H_S(t) - H_{K_S}(-m + t).$$

Since S is aCM, the dual of the resolution of S is a resolution for $K_S(n + 1)$, see [21], Remark 1.4.8. Hence one can write the Hilbert function of the canonical module in terms of the h -vector of S :

$$\Delta^{d+1} H_{K_S}(t) = h_S(d + 1 - t).$$

Therefore by (2) we get

$$\Delta^{d+1} H_Y(t) = \Delta^{d+1} H_S(t) - \Delta^{d+1} H_{K_S}(-m + t) = h_S(t) - h_S(d + 1 - t + m),$$

and for any integer $t > 0$:

$$\begin{aligned} h_Y(t) &= \Delta^d H_Y(t) = \sum_{i=0}^t h_S(i) - \sum_{i=0}^t h_S(d + 1 - i + m) = \\ &= \Delta^d H_S(t) + \sum_{j=0}^{m-t+d} h_S(j) - \sum_{j=0}^{m-t+d} h_S(j) - \sum_{j=m-t+d+1}^{m+d+1} h_S(j) = \\ &= \Delta^d H_S(t) + \Delta^d H_S(m - t + d) - \Delta^d H_S(m + d + 1). \end{aligned}$$

Here $\Delta^d H_S$ is the Hilbert function of the d^{th} -general hyperplane section of S , which is a set of points. Set $r := \min\{i | \Delta^d H_S(i) = \deg S\}$. Since $m \geq r - d$, we obtain that

$$h_Y(t) = \Delta^d H_S(t) + \Delta^d H_S(m - t + d) - \deg S.$$

If $m \geq 2r - d$, then the h -vector of Y is:

$$h_Y(t) = \begin{cases} \Delta^d H_S(t), & t \leq r \\ \deg S, & r \leq t \leq m - r + d \\ \Delta^d H_S(m - t + d), & t \geq m - r + d \end{cases}$$

Notice that $r = \text{reg}(S) - 1$. This follows from Remark 1.2 and from the fact that $r = r(S) + d$, where $r(S)$ is the index of regularity of S .

To prove that h_Y is of decreasing type, we will see that if $\Delta h_Y(j_0) < 0$ for some j_0 , then $\Delta h_Y(j) < 0$ for any $j \geq j_0$. For $t \leq r$ we have $\Delta h_Y(t) = \Delta^{d+1} H_S(t)$ which is strictly positive because it is the h -vector of an aCM scheme. For $r \leq t \leq m - r + d$ we have $\Delta h_Y(t) = 0$, and for any $t \geq m - r + d$ we have $\Delta h_Y(t) = \Delta^d H_S(m - t + d) - \Delta^d H_S(m - t + 1 + d) = -\Delta^{d+1} H_S(m - t + 1 + d) < 0$. So h_Y is of decreasing type. \square

Remark 3.3. (i) From a result of Boij (see [4]), it follows that for $m \gg 0$ any aG divisor on an aCM scheme S is linearly equivalent to $mH - K$, as in Theorem 3.2.

(ii) If we omit the hypothesis of irreducibility for S in Theorem 3.2, we cannot say anything about the irreducibility of its twisted anti-canonical divisors.

(iii) Notice that $m \geq 2r - d$ implies that the h -vectors of the aG schemes Y in Theorem 3.2 have length $m + d$.

(iv) We were not able to compute an upper bound for the regularity of ω_S^\vee in terms of invariants of S such as its Betti numbers. However, using a computer algebra system, such as CoCoA or Macaulay2, it is possible to compute this regularity in concrete examples. In Examples 3.6 and 3.7 we compute this bound for two concrete cases.

Now we use the results of Section 2 and Theorem 3.2 to obtain aG irreducible schemes as divisors on standard determinantal schemes and to determine their Hilbert functions in terms of the degree matrix.

Corollary 3.4. *Let $U = (u_{ij})$ be a degree matrix of size $l \times (l + c - 1)$. Let n be an integer such that $1 \leq c \leq n - 2$. Suppose that $u_{i,i+c} > 0$. Then for $m \geq \max\{2r - n + c, \text{reg}(\omega_S^\vee)\}$, there exists an irreducible and reduced arithmetically Gorenstein $Y \subset \mathbf{P}^n$ of codimension $c + 1$ with the h -vector given by*

(1) for $t < r$,

$$h_Y(t) = \sum_{(V_1|V_2|\dots) \subset U} (-1)^{1+\dim V_2+\dots} \binom{t+c-\text{tr}(V_1|V_2|\dots)}{c};$$

(2) for $r \leq t \leq m - r + n - c$,

$$h_Y(t) = \frac{1}{c!} \sum_{(V_1|V_2|\dots) \subset U} (-1)^{1+\dim V_2+\dots+n} (\text{tr}(V_1|V_2|\dots) - 1) \cdot \dots \cdot (\text{tr}(V_1|V_2|\dots) - c);$$

(3) for $t \geq m - r + n - c$,

$$h_Y(t) = \sum_{(V_1|V_2|\dots) \subset U} (-1)^{1+\dim V_2+\dots} \binom{m-t+n-\text{tr}(V_1|V_2|\dots)}{c}$$

where $r = \text{tr}(V_1^0|V_2^0|\dots|V_c^0) - c$, V_1^0 is the submatrix of U formed by the first l columns and $V_{k+1}^0 = (u_{l,l+k})$ for $k = 1, \dots, c - 1$.

Proof. We can choose a homogeneous matrix A with degree matrix U such that it defines a standard determinantal scheme. From a theorem of Trung ([24], Theorem 4.4 - notice that, since the polynomial ring R is normal, then normality carries on to the reduced, irreducible lifting), there exists a normal, reduced, standard determinantal scheme $S \subset \mathbf{P}^n$ of dimension $d = n - c$ (so $d \geq 2$) with associated degree matrix U . We remark that the

aCM scheme S satisfies property G_1 . Indeed, by Serre's Criterion (see Theorem 11.5 of [9]), S satisfies property G_1 because it is irreducible and normal.

By Theorem 3.2, $r = \text{reg}(S) - 1$, and by Proposition 2.4, $\text{reg}(S) = \text{tr}(V_1^0|V_2^0|\dots|V_c^0) - c + 1$. Hence $r = \text{tr}(V_1^0|V_2^0|\dots|V_c^0) - c$.

Next, we compute $\Delta^d H_S(t)$. By Proposition 2.4, we have that the d -th difference of H_S is given by

$$\begin{aligned} \Delta^d H_S(t) &= \binom{t+n-d}{n-d} + \sum_{(V_1|V_2|\dots) \subset U} (-1)^{1+\dim V_2+\dots} \binom{t+n-\text{tr}(V_1|V_2|\dots)-d}{n-d} \\ &= \binom{t+c}{c} + \sum_{(V_1|V_2|\dots) \subset U} (-1)^{1+\dim V_2+\dots} \binom{t+c-\text{tr}(V_1|V_2|\dots)}{c}. \end{aligned}$$

The Corollary follows from Theorem 3.2 and the expression of $\text{deg}(S)$ obtained in Proposition 2.4. \square

Under stricter numerical conditions on the degree matrix U , we also obtain Hilbert functions of smooth arithmetically Gorenstein schemes.

Corollary 3.5. *Let $U = (u_{ij})$ be a degree matrix of a homogeneous $l \times (l+c-1)$ -matrix of polynomials in $k[x_0, \dots, x_n]$, $1 \leq c \leq n-2$. Assume that $n \leq 2c+1$, $u_{i,j} \neq 0$ for all i, j , $u_{1,k} > 0$ if $k + \lfloor \frac{n-c}{2} \rfloor + 1 - n \leq 0$, and $u_{k+\lfloor \frac{n-c}{2} \rfloor+1-n,k} > 0$ if $k + \lfloor \frac{n-c}{2} \rfloor + 1 - n > 0$. Then for $m \geq \max\{2r - n + c, \text{reg}(\omega_S^\vee)\}$ there exists a smooth, irreducible, reduced arithmetically Gorenstein subscheme $Y \subset \mathbf{P}^n$ of codimension $c+1$ with the h -vector given by*

(1) for $t < r$,

$$h_Y(t) = \sum_{(V_1|V_2|\dots) \subset U} (-1)^{1+\dim V_2+\dots} \binom{t+c-\text{tr}(V_1|V_2|\dots)}{c};$$

(2) for $r \leq t \leq m - r + n - c$,

$$h_Y(t) = \frac{1}{c!} \sum_{(V_1|V_2|\dots) \subset U} (-1)^{1+\dim V_2+\dots+n} (\text{tr}(V_1|V_2|\dots) - 1) \cdot \dots \cdot (\text{tr}(V_1|V_2|\dots) - c);$$

(3) for $t \geq m - r + n - c$,

$$h_Y(t) = \sum_{(V_1|V_2|\dots) \subset U} (-1)^{1+\dim V_2+\dots} \binom{m-t+n-\text{tr}(V_1|V_2|\dots)}{c}$$

where $r = \text{tr}(V_1^0|V_2^0|\dots|V_c^0) - c$, V_1^0 is the submatrix of U formed by the first l columns and $V_{k+1}^0 = (u_{l,l+k})$ for $k = 1, \dots, c-1$.

Proof. A result of Ein [8], Theorem 2.6, ensures that under the hypotheses of the Corollary, there exists a smooth standard determinantal $S \subset \mathbf{P}^n$ scheme with degree matrix U . Theorem 3.2 applies to this S giving a smooth aG scheme Y (in the proof of Theorem 3.2, if S is smooth then the Bertini Theorem gives that $Y \in |mH - K|$ is also smooth). The computation of the Hilbert function of Y follows as in the proof of Corollary 3.4. \square

Here are two examples of h -vectors of irreducible aG schemes obtained using this technique.

Example 3.6. Consider the irreducible and reduced standard determinantal scheme $S \subset \mathbf{P}^5$ associated to the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 & x_5 \end{pmatrix}$$

It is a rational normal scroll surface of \mathbf{P}^5 of degree 4, whose h -vector is $(1, 3)$. Then the h -vector of a general $Y \in |mH - K|$ has form $(1, 4, 4, \dots, 4, 1)$ and has length $m + 2$. The Castelnuovo-Mumford regularity of the dual of the canonical sheaf of the surface equals 5. Then Y is irreducible for $m \geq 5$.

Let d be the degree of a reduced and irreducible subscheme of \mathbf{P}^n of codimension 3. Then we may obtain for Y as in Corollary 3.4 a Gorenstein, codimension 4 h -vector containing an arbitrarily long constant sequence of d 's in the middle.

Example 3.7. Consider the degree matrix

$$\begin{pmatrix} 2 & 2 & 2 & 1 \\ 3 & 3 & 3 & 2 \end{pmatrix}$$

associated to the irreducible curve of \mathbf{P}^4 whose defining matrix is

$$\begin{pmatrix} x_0^2 & x_1^2 & x_2^2 & x_3 \\ x_4^3 & x_0^3 & x_1^3 & x_3^2 \end{pmatrix}$$

The curve has a reduced irreducible lifting $S \subset \mathbf{P}^5$, whose h -vector is $(1, 3, 6, 10, 12, 9, 4, 1)$. The h -vector of a general divisor Y on S linearly equivalent to $mH - K$ has the form $(1, 4, 10, 20, 32, 41, 45, 46, \dots, 46, 45, 41, 32, 20, 10, 4, 1)$, and has length $m + 2$. The Castelnuovo-Mumford regularity of the dual of the canonical sheaf of the surface equals 8. We also need $m \geq 2r - d = 15$. Then Y is an irreducible codimension 4 arithmetically Gorenstein scheme for $m \geq 15$.

APPENDIX

We give here the proofs of the folklore results of Section 2. For the notation, see Section 2.

Proof of Proposition 2.4. To start the computation of the Hilbert function of a standard determinantal scheme $S \subset \mathbf{P}^n$, recall that a minimal free resolution of R/I_S is given by the Eagon-Northcott complex (see [9], Corollary A2.12):

$$\begin{aligned} M_* : 0 \longrightarrow \bigwedge^{l+c-1} F \otimes S_{c-1}(G)^* \otimes \bigwedge^l G^* \longrightarrow \cdots \longrightarrow \bigwedge^{l+1} F \otimes S_1(G)^* \otimes \bigwedge^l G^* \\ \longrightarrow \bigwedge^l F \otimes S_0(G)^* \otimes \bigwedge^l G^* \longrightarrow R \longrightarrow R/I_S \longrightarrow 0, \end{aligned}$$

where $\wedge(P)$, $S(P)$, and P^* mean the exterior algebra, the symmetric algebra, and respectively, the dual of P over R for any R -module P . Using the fact that $\wedge(P \oplus P') =$

$\wedge(P) \otimes \wedge(P')$ and $S(P \oplus P') = S(P) \otimes S(P')$, we get for $i \geq 0$,

$$\begin{aligned}
 M_{i+1} &= \bigwedge^{l+i} F \otimes S_i(G)^* \otimes \bigwedge^l G^* \\
 &= \bigwedge^{l+i} \left(\bigoplus_{1 \leq j \leq l+c-1} R(a_j) \right) \otimes S_i \left(\bigoplus_{1 \leq k \leq l} R(b_k) \right)^* \otimes \bigwedge^l \left(\bigoplus_{1 \leq k \leq l} R(b_k) \right)^* \\
 &\cong \left(\bigoplus_{1 \leq j_1 < \dots < j_{l+i} \leq l+c-1} R(a_{j_1} + \dots + a_{j_{l+i}}) \right) \otimes \left(\bigoplus_{1 \leq k_1 \leq \dots \leq k_i \leq l} R(-b_{k_1} - \dots - b_{k_i}) \right) \otimes \\
 &\otimes R(-b_1 - \dots - b_l) \\
 &\cong \bigoplus_{\substack{1 \leq k_1 \leq \dots \leq k_i \leq l \\ 1 \leq j_1 < \dots < j_{l+i} \leq l+c-1}} R(a_{j_1} + \dots + a_{j_{l+i}} - b_{k_1} - \dots - b_{k_i} - b_1 - \dots - b_l) \\
 &\cong \bigoplus_{1 \leq j_1 < \dots < j_{l+i} \leq l+c-1} R(-u_{1,j_1} - \dots - u_{l,j_l} - u_{k_1,j_{l+1}} - \dots - u_{k_i,j_{l+i}}).
 \end{aligned}$$

By (1), after repeatedly replacing indices of the u_{ij} by some permutations of them, we can write, for $i \geq 0$,

$$(3) \quad M_{i+1} \cong \bigoplus_{(V_1|V_2|\dots) \subset U, \dim V_2 + \dots = i} R(-\text{tr}(V_1|V_2|\dots))$$

Thus, for $i \geq 0$, we have

$$H_{M_{i+1}}(t) = \sum_{(V_1|V_2|\dots) \subset U, \dim V_2 + \dots = i} \binom{t+n - \text{tr}(V_1|V_2|\dots)}{n}.$$

Therefore,

$$\begin{aligned}
 H_S(t) &= \sum_{0 \leq i \leq c} (-1)^i H_{M_i}(t) \\
 &= \binom{t+n}{n} + \sum_{(V_1|V_2|\dots) \subset U} (-1)^{1+\dim V_2+\dots} \binom{t+n - \text{tr}(V_1|V_2|\dots)}{n}.
 \end{aligned}$$

At this point we may simplify. First, since S has codimension c , the coefficient of t^i in $H_S(t)$ has to vanish for $d+1 \leq i \leq n$. Thus, we get

$$s_{n-i}(-1, \dots, -n) + \sum_{(V_1|V_2|\dots) \subset U} (-1)^{1+\dim V_2+\dots} s_{n-i}(\text{tr}(V_1|V_2|\dots) - 1, \dots, \text{tr}(V_1|V_2|\dots) - n) = 0,$$

where s_j are the elementary symmetric functions in n variables. Similarly, since the d -th difference of the Hilbert function of S

$$\Delta^d H_S(t) = \binom{t+c}{c} + \sum_{(V_1|V_2|\dots) \subset U} (-1)^{1+\dim V_2+\dots} \binom{t+c - \text{tr}(V_1|V_2|\dots)}{c}$$

is the Hilbert function of a zero-scheme in \mathbf{P}^c , for $1 \leq j \leq c$ we have that

$$s'_{c-j}(-1, \dots, -c) + \sum_{(V_1|V_2|\dots) \subset U} (-1)^{1+\dim V_2+\dots} s'_{c-j}(\operatorname{tr}(V_1|V_2|\dots) - 1, \dots, \operatorname{tr}(V_1|V_2|\dots) - c) = 0,$$

where s'_j are the elementary symmetric functions in c variables.

Let us call r the length of the h -vector of S or, equivalently,

$$r := \min\{i \mid \Delta^d H_S(i) = \deg S\}$$

(see the end of the proof for an expression of r in terms of the degree matrix U). Then, by the vanishing formulas above, we have that for $t \geq r$,

$$\begin{aligned} \Delta^d H_S(t) &= \deg S \\ &= 1 + \frac{1}{c!} \sum_{(V_1|V_2|\dots) \subset U} (-1)^{c+1+\dim V_2+\dots} s'_c(\operatorname{tr}(V_1|V_2|\dots) - 1, \dots, \operatorname{tr}(V_1|V_2|\dots) - c) \\ &= 1 + \frac{1}{c!} \sum_{(V_1|V_2|\dots) \subset U} (-1)^{c+1+\dim V_2+\dots} (\operatorname{tr}(V_1|V_2|\dots) - 1) \cdots (\operatorname{tr}(V_1|V_2|\dots) - c). \end{aligned}$$

In order to obtain the expression about the regularity of S we look again at the Eagon-Northcott resolution of R/I_S . Notice that, since I_S is a perfect ideal of codimension c , the $\max_{i,j}\{a_{i,j} - i\}$ will be achieved in the last free module of a minimal free resolution of I_S as an R -module. Using the isomorphisms (3), Remark 1.2 and the last observation, we get:

$$\begin{aligned} \operatorname{reg}(S) &= \max\{\operatorname{tr}(V_1|V_2|\dots) - (\dim V_2 + \dim V_3 + \dots)\} = \\ &= \max_i \left\{ \max_{\dim V_2+\dots=i} \{\operatorname{tr}(V_1|V_2|\dots) - i\} \right\} = \max_{\dim V_2+\dots=c-1} \{\operatorname{tr}(V_1|V_2|\dots) - (c-1)\}. \end{aligned}$$

From the way the $u_{i,j}$'s are ordered, we get that the maximum is achieved by the combination of submatrices $W_{c-1} = (V_1^0|V_2^0|\dots|V_c^0)$. The formula for the regularity follows.

Notice that the r defined above satisfies $r = r(S) + d$ where $r(S)$ is the index of regularity of S (see Remark 1.2). Moreover, since S is aCM, $r(S) = \operatorname{reg}(S) - d - 1$ (see Remark 1.2), so

$$r = \operatorname{reg}(S) - 1 = \operatorname{tr}(V_1^0|V_2^0|\dots|V_c^0) - c.$$

□

Proof of Proposition 2.7. The idea of the proof is to see that there exists a reduced standard determinantal scheme $S \subset \mathbf{P}^m$, $m = \max\{n, 2(l-1) + c - 2\}$, with degree matrix U . Then, taking $m - n$ general hyperplane sections of it we get the desired reduced scheme $X \subset \mathbf{P}^n$.

In what follows, whenever a claim involves a *general form* G of degree d in $R' = k[x_0, \dots, x_m]$, it should be understood that the claim is true for all G outside a proper closed subset of the linear system $|\mathcal{O}_{\mathbf{P}^m}(d)|$.

We consider the matrix

$$A = \begin{pmatrix} G_1^1 & \cdots & G_1^c & 0 & 0 & \cdots \\ 0 & G_2^2 & \cdots & G_2^{c+1} & 0 & \cdots \\ & & \ddots & & \ddots & \\ 0 & 0 & \cdots & G_l^l & \cdots & G_l^{l+c-1} \end{pmatrix}$$

where G_i^j are general forms of degree $u_{i,j}$ in $k[x_0, \dots, x_m]$, $m = \max\{n, 2(l-1) + c - 2\}$. We are going to see that the standard determinantal scheme $S \subset \mathbf{P}^m$ defined by the maximal minors of A is a reduced scheme.

We proceed by induction on c .

If $c = 1$, then $I_S = G_1^1 \cdot \dots \cdot G_l^1$ and S is the union of l general hypersurfaces, so S is reduced.

If $c = 2$, it follows from a result of Gaeta ([10]). In this case S is a union of reduced complete intersections.

If $c \geq 3$, we will proceed by induction on l .

When $l = 1$, S is the complete intersection (G_1^1, \dots, G_1^c) , so S is reduced because G_1^j are general.

When $l > 1$, we claim that

Claim 1: $I_S = \bigcap_{i=1}^l ((G_i^i) + I(B_i)) + G_l^{l+c-1} I_Y$ where I_Y is the ideal generated by the maximal minors of the first $l-1$ rows and first $l+c-2$ columns of A and $I(B_i)$ is the ideal of maximal minors of the following $(l-i+1) \times (l+c-2-i)$ submatrix of A :

$$B_i = \begin{pmatrix} G_i^{i+1} & \cdots & G_i^{i+c-1} & 0 & \cdots & \cdots & 0 \\ G_{i+1}^{i+1} & \cdots & & G_{i+1}^{i+c} & 0 & \cdots & 0 \\ & & \ddots & & & \ddots & \\ & & & & & & \\ 0 & \cdots & 0 & G_{l-1}^{l-1} & \cdots & \cdots & G_{l-1}^{l+c-2} \\ 0 & \cdots & \cdots & 0 & G_l^l & \cdots & G_l^{l+c-2} \end{pmatrix}.$$

Proof of claim 1: It is not difficult to check that $I(B_{i+1}) + (G_{i+1}^{i+1}) \supseteq I(B_i)$. Hence, using the modular law and that the G_i^i are general forms, we get that

$$\bigcap_{i=1}^l ((G_i^i) + I(B_i)) + G_l^{l+c-1} I_Y = I(B_1) + \sum_{i=1}^{l-1} G_1^1 \cdot G_2^2 \cdot \dots \cdot G_i^i \cdot I(B_{i+1}) + G_1^1 \cdot G_2^2 \cdot \dots \cdot G_l^l + G_l^{l+c-1} I_Y.$$

This last ideal is the ideal generated by the maximal minors of A . Indeed, the maximal minors of A that contain the last column generate $G_l^{l+c-1} I_Y$. We restrict now to the minors that do not contain the last column. Among them, the minors that do

not contain the first column generate $I(B_1)$; if we consider the ones that contain the first column, then we must distinguish between the ones that do not contain the second column, these generate $G_1^1 I(B_2)$, and the ones that contain the second column: for these we distinguish between the minors that do not contain the third column, these generate $G_1^1 G_2^2 I(B_3)$, and the ones that contain the third column, and so on. Hence

$$I(B_1) + \sum_{i=1}^{l-1} G_1^1 \cdot G_2^2 \cdot \dots \cdot G_i^i \cdot I(B_{i+1}) + G_1^1 \cdot G_2^2 \cdot \dots \cdot G_l^l + G_l^{l+c-1} I_Y = I_S$$

and the claim is proved.

By induction on l , we know that Y is a reduced standard determinantal scheme of codimension c . Moreover, each B_i defines a standard determinantal scheme X_i of codimension $c - 2$.

Claim 2: B_i defines a reduced standard determinantal scheme X_i .

Proof of claim 2: By induction hypothesis we have that

$$C_i = \begin{pmatrix} G_i^{i+1} & \dots & G_i^{i+c-2} & 0 & \dots & \dots & 0 \\ 0 & G_{i+1}^{i+1} & \dots & G_{i+1}^{i+c-1} & 0 & \dots & 0 \\ & & \ddots & & & \ddots & \\ 0 & \dots & 0 & G_{l-1}^{l-1} & \dots & G_{l-1}^{l+c-3} & 0 \\ 0 & \dots & \dots & 0 & G_l^l & \dots & G_l^{l+c-2} \end{pmatrix}$$

is associated to a reduced standard determinantal scheme of codimension $c - 2$. Moreover, if we denote by $I(B_i)$, $I(C_i)$ the ideals generated by the maximal minors of B_i , C_i respectively, we have that

$$\begin{aligned} I(C_i) + (G_{i+1}^{i+1}, \dots, G_l^l, G_i^{i+c-1}, G_{i+1}^{i+c}, \dots, G_{l-1}^{l+c-2}) = \\ I(B_i) + (G_{i+1}^{i+1}, \dots, G_l^l, G_i^{i+c-1}, G_{i+1}^{i+c}, \dots, G_{l-1}^{l+c-2}). \end{aligned}$$

If we call $R' = k[x_0, \dots, x_m]$, we observe that $G_{i+1}^{i+1}, \dots, G_l^l, G_i^{i+c-1}, \dots, G_{l-1}^{l+c-2}$ is an $R'/I(C_i)$ -regular sequence (since this is a regular sequence in R' , $\dim R'/I(C_i) = m - c + 3 \geq 2(l - 1) + 1$, and $G_{i+1}^{i+1}, \dots, G_l^l, G_i^{i+c-1}, \dots, G_{l-1}^{l+c-2}$ do not appear in $I(C_i)$). Hence, the ideal $I(C_i) + (G_{i+1}^{i+1}, \dots, G_l^l, G_i^{i+c-1}, \dots, G_{l-1}^{l+c-2})$ defines a reduced, standard determinantal scheme of codimension $c - 2$.

Then $I(B_i) + (G_{i+1}^{i+1}, \dots, G_l^l, G_i^{i+c-1}, \dots, G_{l-1}^{l+c-2})$ also defines a reduced, aCM scheme of codimension $c - 2$. Since $G_{i+1}^{i+1}, \dots, G_l^l, G_i^{i+c-1}, \dots, G_{l-1}^{l+c-2}$ is a regular sequence modulo $I(B_i)$, we have that X_i is a reduced scheme of codimension $c - 2$ (see [6]). This is the end of the proof of Claim 2.

Therefore,

$$S = Y \cup \bigcup_{i=1}^l X_i \cap G_i^i \cap G_l^{l+c-1},$$

and since Y and $X_i \cap G_i^j \cap G_l^{l+c-1}$ are reduced, S is also a reduced standard determinantal scheme of codimension c in \mathbf{P}^m . Here, we call both G_i^j the form and the hypersurface defined by the form.

Now to obtain the desired reduced standard determinantal subscheme $X \subset \mathbf{P}^n$ we only need to take $m - n$ general hyperplane sections of S . Notice that $m - n \geq m - c = \dim S$, so we are taking general hyperplane sections of a reduced scheme and reducibility is preserved.

□

REFERENCES

- [1] E. ARBARELLO, M. CORNALBA, P.A. GRIFFITHS, J. HARRIS, *Geometry of algebraic curves, vol. I*, Springer-Verlag, Grundlehren der mathematischen Wissenschaften 267 (1985)
- [2] C. BIVIÀ AUSINA, J.J. NUÑO BALLESTEROS, *The Hilbert series of a determinantal ring defined by maximal minors*, preprint, <http://home.imf.au.dk/esn/preprints/099>
- [3] C. BIVIÀ AUSINA, J.J. NUÑO BALLESTEROS, *Multiplicity of iterated Jacobian extensions of weighted homogeneous map germs*, Hokkaido Math. J. **29** (2000), no. 2, 341-368
- [4] M. BOIJ, *Gorenstein Artin algebras and points in projective space*, Bull. London Math. Soc. **31** (1999), 11-16
- [5] W. BRUNS, J. HERZOG, *Cohen-Macaulay Rings*, Cambridge University Press (1993)
- [6] M. P. CAVALIERE, G. NIESI, *On Serre's condition in the form ring of an ideal*, J. Math. Kyoto Univ. **21-3** (1981), 537-546
- [7] E. DE NEGRI, G. VALLA, *The h-vector of a Gorenstein codimension three domain*, Nagoya Math. J. **138**, (1995), 113-140
- [8] L. EIN, *On the cohomology of projective Cohen-Macaulay determinantal subvarieties of \mathbf{P}^n* , in Geometry of complex projective varieties (Cetraro, 1990), 143-152, Seminars and Conferences **9**, Mediterranean Press, 1993
- [9] D. EISENBUD, *Commutative Algebra with a View Toward Algebraic Geometry*, Springer-Verlag, Graduate Texts in Mathematics 150 (1995)
- [10] G. GAETA, *Nuove ricerche sulle curve sghembe algebriche di residuale finito e sui gruppi di punti del piano*, An. Mat. Pura Appl.(4) **31** (1950), 1-64
- [11] A.V. GERAMITA, P. MAROSCIA, L.G. ROBERTS, *The Hilbert function of a reduced k -algebra*, Journ. London Math. Soc. (2) **28**, (1983), 443-452
- [12] A.V. GERAMITA, J.C. MIGLIORE, *Hyperplane sections of a smooth curve in \mathbf{P}^3* , Comm. in Alg. **17**, (1989), 3129-3164
- [13] J. HARRIS, *Curves in Projective Space*, Séminaire de Mathématiques Supérieures **85** (1982), Presses de l'Université de Montréal, Montréal, Que.
- [14] R. HARTSHORNE, *Algebraic Geometry*, Graduate Texts in Mathematics 52, Springer-Verlag (1977)
- [15] R. HARTSHORNE, *Generalized divisors on Gorenstein schemes*, K-Theory **8**, (1994), 287-339
- [16] J. HERZOG, N.V. TRUNG, G. VALLA, *On hyperplane sections of reduced irreducible varieties of low codimension*, J. Math. Kyoto Univ. **34-1** (1994), 47-72
- [17] J.P. JOUANLOU, *Théorèmes de Bertini et applications*, Birkhäuser, Progress in Mathematics 42 (1983)
- [18] J. KLEPPE, J. MIGLIORE, R.M. MIRÓ-ROIG, U. NAGEL AND C. PETERSON, *Gorenstein Liaison, Complete Intersection Liaison Invariants and Unobstructedness*, Memoirs A.M.S., **732** (2001)
- [19] M. KREUZER, *On 0-dimensional complete intersections*, Math. Ann. 292 (1992), no. 1, 43-58
- [20] F.S. MACAULAY, *Some properties of enumeration in the theory of modular systems*, Proc. London Math. Soc. **26**, (1927), 531-555
- [21] J.C. MIGLIORE, *Introduction to Liaison Theory and Deficiency Modules*, Progress in Mathematics 165, Birkhäuser (1998)

- [22] D. MUMFORD, *Lectures on Curves on an Algebraic Surface*, Annals of Mathematics Studies **59**, Princeton University Press (1966)
- [23] R.P. STANLEY, *On the Hilbert Function of Graded Algebras*, Adv. in Mathematics **28** (1978), 57-83
- [24] N.V. TRUNG, *The lifting of determinantal prime ideals*, Manuscripta Math. **91** (1996) no.4, 467-481
- [25] G. VALLA, *Problems and results on the Hilbert Functions of Graded Algebras, Six Lectures on Commutative Algebra*, Progress in Mathematics **166**, Birkhauser Verlag (1998).

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