Gorenstein Biliaison and ACM Sheaves

Marta Casanellas*
Robin Hartshorne

Abstract

Let X be a normal arithmetically Gorenstein scheme in $\mathbb{P}^n$. We give a criterion for all codimension two ACM subschemes of X to be in the same Gorenstein biliaison class on X, in terms of the category of ACM sheaves on X. These are sheaves that correspond to the graded maximal Cohen–Macaulay modules on the homogeneous coordinate ring of X. Using known results on MCM modules, we are able to determine the Gorenstein biliaison classes of codimension two subschemes of certain varieties, including the nonsingular quadric surface in $\mathbb{P}^3$, and the cone over it in $\mathbb{P}^4$. As an application we obtain a new proof of some theorems of Lesperance about curves in $\mathbb{P}^4$, and answer some questions he raised.

0 Introduction

Liaison has become an established technique in algebraic geometry. See, for example, the excellent book of Migliore [14] for an introduction and more than 200 references.

More recently many people have studied Gorenstein liaison and the important open question, whether every arithmetically Cohen–Macaulay (ACM) subscheme of $\mathbb{P}^n$ is in the Gorenstein liaison class of a complete intersection (see, for example, [10]).

In this paper we study Gorenstein biliaison, defined as the equivalence relation generated by elementary Gorenstein biliaisons (see [14, 5.4.7], [9, 1.1], [7, §3]) on a normal arithmetically Cohen–Macaulay (ACM) projective scheme X. We give a new approach to biliaison by relating it to the category of ACM sheaves on X. Note in this paper biliaison is not a synonym for even liaison.

Our main theorems (4.2), (4.3) give a criterion for all codimension two ACM subschemes of X to be in the same Gorenstein biliaison class, in terms of two conditions on the category of arithmetically Cohen–Macaulay (ACM) sheaves on X. These sheaves correspond to maximal

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Cohen–Macaulay (MCM) modules on the homogeneous coordinate ring S of X. Theorem (4.7) gives a criterion for the biliaison class of curves on a three-fold to be determined by their Rao modules. Then making use of previous work on MCM modules over local rings (see for example the book of Yoshino [18]) we are able to conclude results about Gorenstein biliaison of points and curves on certain projective surfaces and three-folds. In particular, we show that all zero-schemes on a non-singular quadric surface in $\mathbb{P}^3$ (5.1) or on a cubic scroll in $\mathbb{P}^4$ (5.4) are in the same Gorenstein biliaison class. On the cone in $\mathbb{P}^4$ over a non-singular quadric surface in $\mathbb{P}^3$, we show that two curves are in the same Gorenstein biliaison class if and only if their Rao modules are isomorphic (6.2). This allows us to give a new proof and extend a theorem of Lesperance concerning biliaison of curves that are disjoint unions of two plane curves in $\mathbb{P}^4$ (6.4), (6.5).

Our hope is that a better understanding of biliaison of codimension two subschemes of hypersurfaces will lead to more insight into the problems of biliaison of codimension three subschemes of projective space. In a separate article [4] we discuss Gorenstein liaison (as opposed to biliaison) and give an analogous criterion in terms of ACM sheaves for all codimension two ACM schemes to be in the same G-liaison class.

Sections 1 and 2 are introductory. Section 3 contains a criterion for biliaison. Section 4 contains the main theorems and their proofs. Sections 5 and 6 contain applications to particular surfaces and three-folds in projective space.

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1 Biliaison

Let V_1, V_2 be equidimensional closed subschemes of dimension r, without embedded components, of $\mathbb{P}^n_k$, the n-dimensional projective space over an algebraically closed field k.

Definition. A closed subscheme $X \subseteq \mathbb{P}^n_k$ is *arithmetically Cohen–Macaulay* (ACM) if its homogeneous coordinate ring $S(X) = k[x_0, \ldots, x_n]/I_X$ (where I_X is the saturated ideal of X) is a Cohen–Macaulay ring. This is equivalent to saying $H^1_*(\mathcal{I}_{X,\mathbb{P}^n}) = 0$ and $H^i_*(\mathcal{O}_X) = 0$ for $0 < i < \dim X$. (For any coherent sheaf $\mathcal{F}$ we denote by $H^i_*(\mathcal{F})$ the sum $\oplus_{\ell \in \mathbb{Z}} H^i(\mathcal{F}(\ell))$.)

Definition. We say V_2 is obtained by an *elementary biliaison of height m* from V_1 if there exists an ACM scheme $Y \subseteq \mathbb{P}^n$, of dimension $r+1$ containing V_1 and V_2, such that $V_2 \sim V_1 + mH$ on Y. (Here $\sim$ means linear equivalence of divisors on Y in the sense of [6] and [7].) The equivalence relation generated by elementary biliaisons is called simply *biliaison.* If Y is a complete intersection scheme in $\mathbb{P}^n$, we speak of *CI-biliaison.* If Y is ACM and satisfies G_0 (Gorenstein in codimension zero) we speak of *Gorenstein biliaison* or *G-biliaison.* If V_1, V_2, Y

are contained in some projective scheme $X \subseteq \mathbb{P}^n$, we speak of *biliaison* (resp. *CI-biliaison*, *G-biliaison*) *on* X.

Note that CI-biliaison in $\mathbb{P}^n$ is equivalent to even CI-liaison (in the sense of [14, 5.1.2]) [6, 4.4]. However, on a projective scheme X, the notions of CI-biliaison, G-biliaison, and even G-liaison on X (in the sense of [14, 5.1.2]) are all three distinct. So we emphasize that the word "biliaison" is not a synonym for "even liaison". Recall, however, that every G-biliaison is an even G-liaison [7, 3.6].

Examples 1.1. If X is a non-singular quadric surface in $\mathbb{P}^3$, then CI-biliaison of zero-schemes on X preserves parity of the length of the zero-scheme. But on a line in X one can make a biliaison from one point to two points. So CI-biliaison and G-biliaison are not equivalent.

If X is a non-singular quadric three-fold in $\mathbb{P}^4$, then Pic $X = \mathbb{Z}$, so every surface in X is a complete intersection. Thus CI-biliaison and G-biliaison coincide on X. But if one takes a rational quartic curve C in X, and a line L in X meeting C in two points, then $C \cup L$ is an arithmetically Gorenstein scheme, so C can be linked to L by one Gorenstein liaison. Since L can be linked to another line L' by another liaison, C and L' are evenly G-linked, but are not equivalent for G-biliaison, since in this case G-biliaison preserves parity of degree.

In this paper we will study biliaison of codimension two subschemes of a normal ACM projective scheme X. Note that the condition of normality implies every divisor on X satisfies G_0, so any biliaison of codimension two subschemes of X is a G-biliaison, and therefore also an even G-liaison [7, 3.6].

For a curve C in $\mathbb{P}^n$, we define as usual the *Rao module* of C to be $M(C) = H^1_*(\mathcal{I}_{C,\mathbb{P}^n}) = \oplus_{\ell \in \mathbb{Z}} H^1(\mathcal{I}_{C,\mathbb{P}^n}(\ell))$. If C is contained in an ACM scheme X of dimension ≥ 2, note that the Rao module $M(C)$ can also be computed as $H^1_*(\mathcal{I}_{C,X})$.

Proposition 1.2. *If curves C, C' in $\mathbb{P}^n$ are equivalent for biliaison, then $M(C') \cong M(C)(h)$ for some integer h.*

Proof. It is enough to check this for one elementary biliaison. So suppose $C' \sim C + mH$ on an ACM surface Y. Then we can compute $M(C) = H^1_*(\mathcal{I}_{C,Y})$ and $M(C') = H^1_*(\mathcal{I}_{C',Y})$. But $\mathcal{I}_{C',Y} \cong \mathcal{I}_{C,Y}(-m)$, so the two Rao modules are isomorphic up to a shift by $-m$.

2 ACM sheaves

Let X be an ACM subscheme of $\mathbb{P}^N$.

Definition. A coherent sheaf $\mathcal{E}$ on X is an ACM *sheaf* if it is locally Cohen–Macaulay on X and $H^i_*(\mathcal{E}) = \oplus_{\ell \in \mathbb{Z}} H^i(X, \mathcal{E}(\ell)) = 0$ for $0 < i < n = \dim X$.

Proposition 2.1. *There is a one-to-one correspondence between* ACM *sheaves on X and graded* MCM *(maximal Cohen–Macaulay) modules on $S(X)$ given by $\mathcal{E} \mapsto H^0_*(\mathcal{E}) = E$ and $E \mapsto E^\sim$.*

Proof. Let $\mathcal{E}$ be an ACM sheaf. Then E is a finitely generated graded $S(X)$-module with $H^0_\mathfrak{m}(E) = H^1_\mathfrak{m}(E) = 0$ by construction, where $\mathfrak{m}$ is the irrelevant ideal of $S(X)$. Furthermore there are isomorphisms $H^i_*(\mathcal{E}) \cong H^{i+1}_\mathfrak{m}(E)$ for $i \geq 1$. Thus E is an MCM module by the local cohomology criterion of depth.

Conversely, if E is graded MCM module, let $\mathcal{E} = E^\sim$ be the associated sheaf. Since E is an MCM module, $\mathcal{E}$ will be a locally CM sheaf, and the same isomorphisms as above show $H^i_*(\mathcal{E}) = 0$ for $0 < i < \dim X$.

Remark 2.2. In the definition of ACM sheaf, we could omit the requirement $\mathcal{E}$ locally CM, if we add a condition $H^0(\mathcal{E}(\ell)) = 0$ for $\ell \ll 0$. Because then $E = H^0_*(\mathcal{E})$ will be a finitely generated module on $S(X)$, the condition $H^i_*(\mathcal{E}) = 0$ for $0 < i < \dim X$ makes it an MCM module, and then $\mathcal{E} = E^\sim$ will be locally CM.

Proposition 2.3. *Let ω be a dualizing sheaf on the* ACM *scheme X. For any coherent sheaf $\mathcal{F}$ denote by $\mathcal{F}^\omega$ the sheaf $\mathcal{H}om(\mathcal{F}, \omega)$. Then*

a) *The functor $\mathcal{E} \mapsto \mathcal{E}^\omega$ is a contravariant, exact functor on the category of locally Cohen–Macaulay sheaves on X.*

b) *For any such $\mathcal{E}$, there is a natural isomorphism $\mathcal{E}^{\omega\omega} \cong \mathcal{E}$.*

c) *Serre duality gives $H^i(\mathcal{E}^\omega)$ dual to $H^{n-i}(\mathcal{E})$ for any such $\mathcal{E}$, where $n = \dim X$.*

d) *$\mathcal{E}$ is* ACM *if and only if $\mathcal{E}^\omega$ is* ACM.

Proof. If $\mathcal{E}$ is locally CM, then by local duality at the various points of X we find $\mathcal{E}xt^i(\mathcal{E}, \omega) = 0$ for all $i > 0$. Thus the spectral sequence of local and global Ext degenerates and we find $\operatorname{Ext}^i(\mathcal{E}, \omega) = H^i(\mathcal{E}^\omega)$ for all i. Serre duality then says $H^i(\mathcal{E}^\omega)$ is dual to $H^{n-i}(\mathcal{E})$, where $n = \dim X$. Furthermore, $\mathcal{E}$ locally CM implies $\mathcal{E}^\omega$ locally CM (repeat the proof of [6, 1.13,1.14] with $\mathcal{E}^\omega$ in place of $\mathcal{E}^\vee$). The functor is exact, because if

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

is an exact sequence of locally CM sheaves, then there is an exact sequence

$$0 \to \mathcal{E}''^\omega \to \mathcal{E}^\omega \to \mathcal{E}'^\omega \to \mathcal{E}xt^1(\mathcal{E}'', \omega) = 0.$$

It is well-known that $\mathcal{E}^{\omega\omega} \cong \mathcal{E}$ (see, for example, [7, 1.5]). Finally, Serre duality shows that $\mathcal{E}$ is ACM if and only if $\mathcal{E}^\omega$ is ACM.

Corollary 2.4. *If X is arithmetically Gorenstein, the dual of an* ACM *sheaf is again an* ACM *sheaf.*

Proof. Indeed, $\mathcal{E}^{\omega}$ will be a twist of $\mathcal{E}^{\vee}$.

Remark 2.5. If Y is a codimension one ACM subscheme of the ACM scheme X, then the ideal sheaf $\mathcal{I}_Y$ of Y on X is an ACM sheaf on X. Conversely, if X is normal and ACM, every rank one ACM sheaf on X is isomorphic to an almost Cartier divisor, and the effective ones correspond to the codimension one ACM subschemes of X [6, 2.7].

Definition. An ACM sheaf $\mathcal{E}$ on a normal ACM scheme X is *layered* if there exists a filtration

$$0 = \mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \cdots \subseteq \mathcal{E}_r = \mathcal{E}$$

whose quotients $\mathcal{E}_i/\mathcal{E}_{i-1}$ are rank 1 ACM sheaves on X for $i = 1, \ldots, r$.

Definition. If $\mathcal{E}$ is a torsion-free sheaf of rank r on a normal scheme X we define its *first Chern class* $c_1(\mathcal{E})$ to be the double dual of the highest exterior power $\Lambda^r\mathcal{E}$ of $\mathcal{E}$. We consider $c_1(\mathcal{E})$ as an element of the group APic X of almost Cartier divisors modulo linear equivalence. We say $\mathcal{E}$ is *orientable* if $c_1(\mathcal{E}) \cong \mathcal{O}_X(\ell)$ is a multiple of the hyperplane class $\mathcal{O}_X(1)$, for some $\ell \in \mathbb{Z}$.

Definition. We say a coherent sheaf $\mathcal{E}$ on X is *dissocié* if $\mathcal{E} \cong \oplus_{i=1}^r \mathcal{O}_X(a_i)$ for some $a_i \in \mathbb{Z}$.

3 A criterion for biliaison

We give a criterion for when two codimension 2 subschemes V, V' of a normal ACM projective scheme X are in the same G-biliaison class.

Theorem 3.1. *Let V, V' be codimension 2 subschemes without embedded components of a normal projective* ACM *scheme X. Then V and V' are in the same G-biliaison equivalence class on X if and only if there exist resolutions on X*

$$0 \to \mathcal{E} \to \mathcal{N} \to \mathcal{I}_V(a) \to 0$$

$$0 \to \mathcal{E}' \to \mathcal{N} \to \mathcal{I}_{V'}(a') \to 0$$

with $a, a' \in \mathbb{Z}$, the same coherent sheaf $\mathcal{N}$ in the middle, and where $\mathcal{E}, \mathcal{E}'$ are layered ACM *sheaves (cf. §2) on X, of the same rank, and the rank 1 factors of the layerings of $\mathcal{E}$ and $\mathcal{E}'$ are isomorphic, up to twist, in some order.*

Proof. First suppose V and V' are equivalent by biliaison. We proceed by induction on the number of elementary biliaisons required.

If V and V' are related by a single elementary biliaison, then there is an ACM divisor Y in X, containing V and V', and $V' \sim V + mH$ on Y for some $m \in \mathbb{Z}$. This gives an isomorphism of ideal sheaves on Y, $\mathcal{I}_{V',Y} \cong \mathcal{I}_{V,Y}(-m)$. Thus we can write

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_Y(-m) \arrow[r] & \mathcal{I}_V(-m) \arrow[r] & \mathcal{I}_{V,Y}(-m) \arrow[r] \arrow[d, equal] & 0 & (1) \\
0 \arrow[r] & \mathcal{I}_Y \arrow[r] & \mathcal{I}_{V'} \arrow[r] & \mathcal{I}_{V',Y} \arrow[r] & 0. & (2)
\end{tikzcd}$$

Let $\mathcal{F}$ be the fibered sum of $\mathcal{I}_V(-m)$ and $\mathcal{I}_{V'}$ over $\mathcal{I}_{V',Y}$. Then we have exact sequences

$$0 \to \mathcal{I}_Y(-m) \to \mathcal{F} \to \mathcal{I}_{V'} \to 0 \tag{3}$$

$$0 \to \mathcal{I}_Y \to \mathcal{F} \to \mathcal{I}_V(-m) \to 0. \tag{4}$$

Thus the condition of the theorem is satisfied, with $\mathcal{E}, \mathcal{E}'$ being the rank 1 ACM sheaves $\mathcal{I}_Y, \mathcal{I}_Y(-m)$.

Now suppose that V' and V'' are related by $r-1$ elementary biliaisons. By the induction step, there are sequences

$$0 \to \mathcal{E}' \to \mathcal{N} \to \mathcal{I}_{V'}(a') \to 0 \tag{5}$$

$$0 \to \mathcal{E}'' \to \mathcal{N} \to \mathcal{I}_{V''}(a'') \to 0 \tag{6}$$

with $\mathcal{E}', \mathcal{E}''$ of rank $r-1$, layered, with isomorphic factors up to twist and order.

Twisting (3) by a' and combining with (5), taking fibered sums as above, we get new sequences

$$0 \to \mathcal{I}_Y(-m+a') \to \mathcal{G} \to \mathcal{N} \to 0 \tag{7}$$

$$0 \to \mathcal{E}' \to \mathcal{G} \to \mathcal{F}(a') \to 0. \tag{8}$$

Now composing with the maps $\mathcal{N} \to \mathcal{I}_{V''}(a'')$ from (6) and $\mathcal{F}(a') \to \mathcal{I}_V(-m+a')$ from (4) and using the snake lemma, we get

$$0 \to \mathcal{H}'' \to \mathcal{G} \to \mathcal{I}_{V''}(a'') \to 0 \tag{9}$$

$$0 \to \mathcal{H} \to \mathcal{G} \to \mathcal{I}_V(-m+a') \to 0 \tag{10}$$

where $\mathcal{H}$ and $\mathcal{H}''$ are extensions

$$0 \to \mathcal{E}' \to \mathcal{H} \to \mathcal{I}_Y(a') \to 0 \tag{11}$$

$$0 \to \mathcal{I}_Y(-m+a') \to \mathcal{H}'' \to \mathcal{E}'' \to 0. \tag{12}$$

Thus $\mathcal{H}, \mathcal{H}''$ are layered ACM sheaves with isomorphic factors, up to twist and order, and so sequences (9), (10) satisfy the condition of the theorem.

For the reverse implication of the theorem, we will need several lemmas.

Lemma 3.2. *Let X be an integral projective scheme satisfying the condition S_2 of Serre. Let $\mathcal{E}$ be a torsion-free coherent sheaf on X, locally free in codimension 1. Let $W \subseteq H^0(\mathcal{E})$ be a subspace, and let $\mathcal{E}_0$ be the subsheaf generated by W. Then the following conditions are equivalent.*

(i) *There is an $s \in W$ such that $\mathcal{E}' = \mathcal{E}/(s)$ is torsion-free and locally free in codimension 1.*

(ii) (a) *for all $x \in X$ of codimension* 1, $\operatorname{rank}(\mathcal{E}_0 \otimes k(x) \overset{\sigma_x}{\to} \mathcal{E} \otimes k(x)) \geq 1$ *and*

(b) *either* $\operatorname{rank} \mathcal{E}_0 \geq 2$ *or $\mathcal{E}_0 \cong \mathcal{O}_X$ and $\mathcal{E}/\mathcal{E}_0$ is torsion-free and locally free in codimension* 1.

Proof. [8, 2.6].

Lemma 3.3 (The case rank $\mathcal{N} = 2$). *With the hypotheses of the theorem suppose given*

$$0 \to \mathcal{L} \to \mathcal{N} \to \mathcal{I}_V(a) \to 0 \tag{13}$$

$$0 \to \mathcal{L}' \to \mathcal{N} \to \mathcal{I}_{V'}(a') \to 0 \tag{14}$$

with $\operatorname{rank} \mathcal{N} = 2$ *and $\mathcal{L}, \mathcal{L}'$* ACM *sheaves of* rank 1, *isomorphic up to twist. Then either $V = V'$ or V and V' are related by a single elementary biliaison on X.*

Proof. Make a diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathcal{L}' \arrow[d] \arrow[r, equal] & \mathcal{L}' \arrow[d, "\alpha"] & \\
0 \arrow[r] & \mathcal{L} \arrow[r] \arrow[d, equal] & \mathcal{N} \arrow[r] \arrow[d] & \mathcal{I}_V(a) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{L} \arrow[r] & \mathcal{I}_{V'}(a') \arrow[r] \arrow[d] & \mathcal{R} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

We start by considering the composed map $\alpha : \mathcal{L}' \to \mathcal{I}_V(a)$. If $\alpha = 0$, then $\mathcal{L}'$ maps to $\mathcal{L}$ with torsion free cokernel, so $\mathcal{L} \cong \mathcal{L}'$, $a = a'$, $V = V'$. Otherwise α must be injective, and we get the diagram shown. Now the map α composed with the inclusion $\mathcal{I}_V(a) \subseteq \mathcal{O}_X(a)$ gives the ideal of an ACM divisor Y on X, so that $\mathcal{L}' = \mathcal{I}_Y(a)$. Then V is contained in Y and $\mathcal{R} \cong \mathcal{I}_{V,Y}(a)$. From the bottom row of the diagram we see also that $\mathcal{L} = \mathcal{I}_Y(a')$ and $\mathcal{R} \cong \mathcal{I}_{V',Y}(a')$. Thus $\mathcal{I}_{V,Y}(a) \cong \mathcal{I}_{V',Y'}(a')$, so $V' \sim V + (a' - a)H$ on Y.

Lemma 3.4. *Let V be a codimension 2 subscheme of the normal* ACM *scheme X and suppose given a sequence*

$$0 \to \mathcal{E} \to \mathcal{N} \to \mathcal{I}_V(a) \to 0$$

where $\mathcal{N}$ is coherent and $\mathcal{E}$ is a layered ACM *sheaf with* rank 1 ACM *factors $\mathcal{L}_i$, $i = 1, \dots, r$. Then for any choice of $b_i \gg 0$ and $s_i \in \operatorname{Hom}(\mathcal{L}_i, \mathcal{N}(b_i))$ sufficiently general, we will get a sequence*

$$0 \to \oplus \mathcal{L}_i(-b_i) \xrightarrow{\oplus s_i} \mathcal{N} \to \mathcal{I}_{V'}(a') \to 0$$

for some codimension 2 subscheme V' of X in the same G-biliaison class as V.

Proof. We will split off the factors of $\mathcal{E}$ one by one. Since $\mathcal{E}$ is layered, we can write

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{L} \to 0$$

where $\mathcal{E}'$ is layered of rank $r - 1$, and $\mathcal{L}$ is one of the factors. We write a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathcal{E}' \arrow[r, phantom, "="] \arrow[d] & \mathcal{E}' \arrow[d] & & \\
0 \arrow[r] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{N} \arrow[r] \arrow[d] & \mathcal{I}_V(a) \arrow[r] \arrow[d, phantom, "\|"] & 0 \\
0 \arrow[r] & \mathcal{L} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{I}_V(a) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

where $\mathcal{F} = \mathcal{N}/\mathcal{E}'$ is of rank 2. Choose $b \gg 0$ large enough so that the sheaf $\mathcal{H}om(\mathcal{L}, \mathcal{N}(b))$ is generated by global sections, and let $W = H^0(\mathcal{H}om(\mathcal{L}, \mathcal{F}(b)))$ be the image of $H^0(\mathcal{H}om(\mathcal{L}, \mathcal{N}(b)))$. There is an exact sequence

$$0 \to \mathcal{H}om(\mathcal{L}, \mathcal{E}') \to \mathcal{H}om(\mathcal{L}, \mathcal{N}) \to \mathcal{H}om(\mathcal{L}, \mathcal{F}) \to \mathcal{E}xt^1(\mathcal{L}, \mathcal{E}') \to \dots .$$

Since X is normal, $\mathcal{L}$ is locally free in codimension 1, so the sheaf $\mathcal{E}xt^1(\mathcal{L}, \mathcal{E}')$ has support in codimension ≥ 2. It follows that W generates a subsheaf $\mathcal{F}_0$ of $\mathcal{H}om(\mathcal{L}, \mathcal{F}(b))$ that is equal to $\mathcal{H}om(\mathcal{L}, \mathcal{F}(b))$ in codimension 1 and therefore has rank $= 2$. Thus we can apply (3.2) and find that for any sufficiently general $s \in \operatorname{Hom}(\mathcal{L}, \mathcal{N}(b))$, the composed map $\mathcal{L}(-b) \to \mathcal{F}$ will have cokernel torsion-free and locally free in codimension 1. Its first Chern class will be $a + b$, so we get a sequence

$$0 \to \mathcal{L}(-b) \xrightarrow{s} \mathcal{F} \to \mathcal{I}_{V'}(a+b) \to 0$$

for some codimension 2 subscheme V'. According to (3.3), V and V' are related by a single biliaison.

Let $\mathcal{N} \to \mathcal{I}_{V'}(a+b)$ be the composed map, and let $\mathcal{R}$ be the kernel. Then we have a new diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathcal{E}' \arrow[r, equal] \arrow[d] & \mathcal{E}' \arrow[d] & & \\
0 \arrow[r] & \mathcal{R} \arrow[r] \arrow[d] & \mathcal{N} \arrow[r] \arrow[d] & \mathcal{I}_{V'}(a+b) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{L}(-b) \arrow[r, "s"] \arrow[d] \arrow[ur, "s"] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{I}_{V'}(a+b) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

and since the map s lifts to $\mathcal{N}$ by construction, and its image in $\mathcal{I}_{V'}(a+b)$ is zero, the first column sequence splits, and we find $\mathcal{R} \cong \mathcal{E}' \oplus \mathcal{L}(-b)$. Furthermore, the map $\mathcal{L}(-b) \to \mathcal{N}$ is the chosen one s.

We repeat this procedure with a factor of $\mathcal{E}'$. Continuing in the same manner, after r steps we obtain the result of the statement, each time replacing V by something in the same biliaison class.

Proof of 3.1*, continued.* Suppose given two subschemes V, V' and sequences

$$0 \to \mathcal{E} \to \mathcal{N} \to \mathcal{I}_V(a) \to 0$$

$$0 \to \mathcal{E}' \to \mathcal{N} \to \mathcal{I}_{V'}(a') \to 0$$

as in the theorem. Here $\mathcal{E}$ and $\mathcal{E}'$ are layered ACM sheaves with factors $\mathcal{L}_1, \ldots, \mathcal{L}_r$ isomorphic up to twist and order. Applying (3.4) to both of them, we obtain

$$0 \to \oplus \mathcal{L}_i(-b_i) \xrightarrow{\oplus s_i} \mathcal{N} \to \mathcal{I}_{V_1}(a_1) \to 0$$

$$0 \to \oplus \mathcal{L}'_i(-b'_i) \xrightarrow{\oplus s'_i} \mathcal{N} \to \mathcal{I}_{V'_1}(a'_1) \to 0$$

where V and V_1 are biliaison equivalent, and V', V'_1 are biliaison equivalent.

But now, since the factors of $\mathcal{E}$ and $\mathcal{E}'$ were isomorphic up to shift and order, we may assume $\mathcal{L}_i = \mathcal{L}'_i$ for each i. Also, we can take the b_i in (3.4) sufficiently large arbitrarily, so we may assume $b_i = b'_i$ for each i. And when we choose the section s'_i, we may assume that they are equal to the s_i. Then $a_1 = a'_1$, $V_1 = V'_1$, and we are done.

4 The main theorem

In this section we will show that the property of ACM codimension 2 subschemes of X all being in the same biliaison class is equivalent to two statements about the category of ACM sheaves on X.

Let X be a normal ACM scheme in some projective space, of dimension at least 2. Our first condition is

(A) Any two ACM subschemes V, V' of X of codimension 2 are equivalent for Gorenstein biliaison.

Our second condition is

(B) Every orientable (cf. §2) ACM sheaf $\mathcal{E}$ on X has a presentation

$$0 \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{E} \to 0$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ are layered ACM sheaves.

To state our third condition we need some notation. Let $\mathcal{M}$ be the category of layered ACM sheaves on X. Let $G(\mathcal{M})$ be the Grothendieck group of the category $\mathcal{M}$, generated by the objects of $\mathcal{M}$ and with relations $\mathcal{F} - \mathcal{F}' - \mathcal{F}''$ whenever there is an exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ in $\mathcal{M}$. We regard $G(\mathcal{M})$ as a $\mathbb{Z}[h]$-module, where $h \cdot \mathcal{F} = \mathcal{F}(1)$ for any $\mathcal{F} \in \mathcal{M}$. We consider the quotient group $G' = G(\mathcal{M})/(1-h)G(\mathcal{M})$. Roughly speaking, G' is the group generated by the rank 1 ACM sheaves on X, with relations coming from exact sequences in $\mathcal{M}$, and where each sheaf is identified with all of its twists. There is a natural homomorphism $c_1 : G(\mathcal{M}) \to \text{APic } X$ obtained by taking the first Chern class. This passes to the quotient to give a map $c_1 : G' \to \text{APic } X/\mathbb{Z} \cdot \mathcal{O}_X(1)$. Our third condition is

(C) If $\mathcal{E}$ is an orientable layered ACM sheaf on X, then its class in G' is equal to $r \cdot \mathcal{O}$, where $r = \text{rank } \mathcal{E}$.

Note that (C) is equivalent to saying the kernel of the map $c_1 : G' \to \text{APic } X/\mathbb{Z} \cdot \mathcal{O}_X(1)$ is just the subgroup $\mathbb{Z} \cdot \mathcal{O}$ of G'.

Our goal is to prove (A) $\Leftrightarrow$ (B) + (C). We split the proof in several parts.

Lemma 4.1. *Let X be a normal* ACM *scheme, and let V, V' be two codimension* 2 *subschemes in the same biliaison class. Suppose also given exact sequences*

$$0 \to \mathcal{E} \to \mathcal{G} \to \mathcal{I}_V \to 0 \tag{1}$$

$$0 \to \mathcal{E}' \to \mathcal{G}' \to \mathcal{I}_{V'} \to 0 \tag{2}$$

with $\mathcal{E}, \mathcal{E}', \mathcal{G}, \mathcal{G}'$ coherent sheaves. Then there are other coherent sheaves $\mathcal{H}, \mathcal{R}, \mathcal{S}$, and layered ACM *sheaves $\mathcal{F}, \mathcal{F}'$ with the same* rank 1 *factors up to order and twist, and exact sequences with suitable twists a, a',*

$$0 \to \mathcal{R} \to \mathcal{H} \to \mathcal{G}(a) \to 0 \tag{3}$$

$$0 \to \mathcal{E}'(a') \to \mathcal{R} \to \mathcal{F} \to 0 \tag{4}$$

$$0 \to \mathcal{S} \to \mathcal{H} \to \mathcal{G}'(a') \to 0 \tag{5}$$

$$0 \to \mathcal{E}(a) \to \mathcal{S} \to \mathcal{F}' \to 0. \tag{6}$$

Proof. By (3.1) there are exact sequences

$$0 \to \mathcal{F} \to \mathcal{N} \to \mathcal{I}_V(a) \to 0 \tag{7}$$

$$0 \to \mathcal{F}' \to \mathcal{N} \to \mathcal{I}_{V'}(a') \to 0 \tag{8}$$

with the same coherent sheaf $\mathcal{N}$ in the middle, and layered ACM sheaves $\mathcal{F}, \mathcal{F}'$ with the same rank 1 factors, up to twist. We make fibered sum constructions, as in the proof of (3.1).

From (1) and (7) we obtain

$$0 \to \mathcal{E}(a) \to \mathcal{A} \to \mathcal{N} \to 0 \tag{9}$$

$$0 \to \mathcal{F} \to \mathcal{A} \to \mathcal{G}(a) \to 0. \tag{10}$$

From (2) and (8) we obtain

$$0 \to \mathcal{E}'(a') \to \mathcal{B} \to \mathcal{N} \to 0 \tag{11}$$

$$0 \to \mathcal{F}' \to \mathcal{B} \to \mathcal{G}'(a') \to 0. \tag{12}$$

Then from (9) and (11) we obtain

$$0 \to \mathcal{E}(a) \to \mathcal{H} \to \mathcal{B} \to 0 \tag{13}$$

$$0 \to \mathcal{E}'(a') \to \mathcal{H} \to \mathcal{A} \to 0. \tag{14}$$

Now compose the maps $\mathcal{H} \to \mathcal{B} \to \mathcal{G}'(a')$ and let the kernel be $\mathcal{S}$, to get (5). Then because of (12) and (13) we get (6). Similarly, comparing $\mathcal{H} \to \mathcal{A} \to \mathcal{G}(a)$ we get (3) and (4).

Proposition 4.2. *Let X be a normal projective* ACM *scheme. Then* (A) $\Rightarrow$ (B) + (C).

Proof. To prove (B), let $\mathcal{E}$ be any orientable ACM sheaf. Then we can find a dissocié sheaf $\mathcal{L}$ and an exact sequence

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{I}_V(b) \to 0$$

for some codimension 2 subscheme of V [8, 1.12]. Furthermore, since $\mathcal{E}$ is an ACM sheaf, V will be an ACM subscheme. By hypothesis (A) there is a complete intersection Z of X with two hypersurfaces in the same biliaison class as V. Let

$$0 \to \mathcal{P}_2 \to \mathcal{P}_1 \to \mathcal{I}_Z \to 0$$

be its resolution, where $\mathcal{P}_i$ are dissocié sheaves.

Now by the lemma, there are sheaves $\mathcal{H}$, $\mathcal{R}$, $\mathcal{S}$, $\mathcal{F}$, $\mathcal{F}'$ and exact sequences (3)–(6) in changed notation. Sequence (3) gives a resolution

$$0 \to \mathcal{R} \to \mathcal{H} \to \mathcal{E}(a-b) \to 0.$$

Then sequence (4) shows that $\mathcal{R}$ is layered, and sequences (5), (6) show that $\mathcal{H}$ is layered. This proves (B).

To prove (C), let $\mathcal{E}$ be an orientable layered ACM sheaf. Let $\mathcal{L}, V, Z, \mathcal{P}_1, \mathcal{P}_2$ be as in the first part of the proof. Then the sequences (3)–(6) of the lemma show that in the quotient Grothendieck group G', we have

$$\mathcal{H} = \mathcal{E} + \mathcal{P}_2 + \mathcal{F} = \mathcal{P}_1 + \mathcal{L} + \mathcal{F}'.$$

Since $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{L}$ are dissocié, and $\mathcal{F}$ and $\mathcal{F}'$ have the same rank 1 factors, up to twist, we find $\mathcal{E} = r \cdot \mathcal{O}$ in G', where $r = \operatorname{rank} \mathcal{E}$.

Theorem 4.3. *Let X be a normal* ACM *projective scheme, and assume furthermore either* a) *X is arithmetically Gorenstein, or* b) APic *X is generated as a monoid by the* rank 1 ACM *sheaves. Then* (B) + (C) $\Rightarrow$ (A).

We will need several lemmas before proving the theorem.

Lemma 4.4. *Under either hypothesis* a) *or* b) *of* (4.3)*, the condition* (B) *implies*

(B*) *Every orientable* ACM *sheaf $\mathcal{E}$ on X has a resolution*

$$0 \to \mathcal{E} \to \mathcal{G}_1 \to \mathcal{G}_2 \to 0$$

where $\mathcal{G}_1$ and $\mathcal{G}_2$ are layered ACM *sheaves.*

Proof. a) If X is arithmetically Gorenstein, then the dual of an orientable ACM sheaf is again an orientable ACM sheaf (2.4). So we apply (B) to $\mathcal{E}^\vee$ and dualize.

b) Suppose that APic X is generated as a monoid by rank 1 ACM sheaves. Then for any ACM sheaf $\mathcal{E}$ we can write $-c_1(\mathcal{E}) = \Sigma \mathcal{L}_i$ in APic X, where the $\mathcal{L}_i$ are rank 1 ACM sheaves. Then $\mathcal{E}' = \mathcal{E} \oplus \bigoplus \mathcal{L}_i$ is an orientable ACM sheaf. Applying (B) to $\mathcal{E}'$ we get

$$0 \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{E}' \to 0$$

with the $\mathcal{F}_i$ layered ACM sheaves. Now composing with the projection $\mathcal{E}' \to \mathcal{E}$ we get

$$0 \to \mathcal{R} \to \mathcal{F}_1 \to \mathcal{E} \to 0$$

where $\mathcal{R}$ is an extension

$$0 \to \mathcal{F}_2 \to \mathcal{R} \to \bigoplus \mathcal{L}_i \to 0.$$

Thus $\mathcal{E}$ has a resolution of the same type, and we have established the stronger condition

(B$'$) Every ACM sheaf $\mathcal{E}$ on X (not necessarily orientable) has a resolution of the type (B).

Now we can apply the functor $\mathcal{E} \mapsto \mathcal{E}^\omega$ of (2.3) to the category of all ACM sheaves and obtain the condition

(B$'^*$) Every ACM sheaf $\mathcal{E}$ on X has a resolution of type (B*).

Since (B$'^*$) obviously implies (B*) we are done.

Lemma 4.5. *Let X be a normal* ACM *projective scheme, and suppose given a codimension two subscheme V and exact sequences*

$$0 \to \mathcal{E}_0 \oplus \mathcal{E}' \to \mathcal{N} \to \mathcal{I}_V(a) \to 0$$

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{F} \to 0$$

with $\mathcal{E}_0$ and $\mathcal{N}$ coherent, and $\mathcal{E}', \mathcal{E}, \mathcal{F}$ layered ACM *sheaves on X. Then for any $b \gg 0$ there exists a sequence*

$$0 \to \mathcal{E}_0 \oplus \mathcal{E}'(-b) \to \mathcal{N} \to \mathcal{I}_{V'}(a') \to 0$$

with V' in the same biliaison class as V, and with the extra property that the map $\mathcal{E}'(-b) \to \mathcal{N}$ in this sequence extends to a map of $\mathcal{E}(-b) \to \mathcal{N}$.

Proof. By induction on the rank of $\mathcal{E}'$. If $\mathcal{E}'$ has rank 1, we can argue as in the proof of (3.4). Consider the sequence of sheaves

$$\mathcal{H}om(\mathcal{E}, \mathcal{N}) \to \mathcal{H}om(\mathcal{E}', \mathcal{N}) \to \mathcal{E}xt^1(\mathcal{F}, \mathcal{N}).$$

Since X is normal, $\mathcal{F}$ is locally free in codimension 1, and so the $\mathcal{E}xt$ sheaf has support in codimension ≥ 2. Take any $b \gg 0$ so that $\mathcal{H}om(\mathcal{E}, \mathcal{N}(b))$ is generated by global sections, and let $W \subseteq H^0(\mathcal{H}om(\mathcal{E}', \mathcal{N}(b)))$ be the image of $H^0(\mathcal{H}om(\mathcal{E}, \mathcal{N}(b))$. Then W satisfies the conditions of (3.2), and so a general $s \in W$ will give the required map of $\mathcal{E}'(-b) \to \mathcal{N}$.

Now suppose rank $\mathcal{E}' \geq 2$. Then we can split off a rank 1 factor $\mathcal{L}$

$$0 \to \mathcal{E}'' \to \mathcal{E}' \to \mathcal{L} \to 0$$

and get

$$0 \to \mathcal{E}'' \to \mathcal{E} \to \mathcal{R} \to 0$$

where

$$0 \to \mathcal{L} \to \mathcal{R} \to \mathcal{F} \to 0.$$

First apply the splitting technique of (3.4) to get

$$0 \to \mathcal{E}_0 \oplus \mathcal{E}'' \oplus \mathcal{L}(-b_1) \to \mathcal{N} \to \mathcal{I}_{V_1}(a_1) \to 0$$

for some $b_1 \gg 0$ and V_1 in the same biliaison class as V. Now apply the induction hypothesis to $\mathcal{E}''$ to find a sequence

$$0 \to \mathcal{E}_0 \oplus \mathcal{E}''(-b_2) \oplus \mathcal{L}(-b_1) \to \mathcal{N} \to \mathcal{I}_{V_2}(a_2) \to 0$$

for some $b_2 \gg 0$, and V_2 in the same biliaison class as V_1, where the map $\mathcal{E}''(-b_2) \to \mathcal{N}$ extends to $\mathcal{E}(-b_2)$. We may assume $b_2 \gg b_1$ and then change the map from $\mathcal{L}(-b_1) \to \mathcal{N}$ again so that $b_1 = b_2 = b$.

Consider the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathcal{E}'' \arrow[r, phantom, "="] \arrow[d] & \mathcal{E}'' \arrow[d] & & \\
0 \arrow[r] & \mathcal{E}' \arrow[r] \arrow[d] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d, phantom, "\|"] & 0 \\
0 \arrow[r] & \mathcal{L} \arrow[r] \arrow[d] & \mathcal{R} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] & 0 \\
 & 0 & 0. & &
\end{tikzcd} \tag{1}$$

Consider also the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
 & \mathcal{E}_0 \oplus \mathcal{E}''(-b) \arrow[r, phantom, "="] \arrow[d] & \mathcal{E}_0 \oplus \mathcal{E}''(-b) \arrow[d] \\
0 \arrow[r] & \mathcal{E}_0 \oplus \mathcal{E}'(-b) \arrow[r] \arrow[d] & \mathcal{N} \arrow[d] \\
 & \mathcal{L}(-b) \arrow[r] \arrow[d] & \mathcal{G} \arrow[d] \\
 & 0 & 0
\end{tikzcd} \tag{2}$$

using the map $\mathcal{E}'(-b)$ to $\mathcal{N}$ obtained by restricting from $\mathcal{E}(-b)$ to $\mathcal{N}$. We obtain a map $s_0 : \mathcal{L}(-b) \to \mathcal{G}$ on the quotients with no special property. (Note rank $\mathcal{G} = 2$.)

Taking $\mathcal{H}om(\cdot, \mathcal{N}(b))$ of the first diagram, and composing with the map $\mathcal{N} \to \mathcal{G}$ from the second diagram, we get

$$\begin{tikzcd}
\mathcal{H}om(\mathcal{R}, \mathcal{N}(b)) \arrow[r] \arrow[d] & \mathcal{H}om(\mathcal{E}, \mathcal{N}(b)) \arrow[d] \\
\mathcal{H}om(\mathcal{L}, \mathcal{N}(b)) \arrow[r] \arrow[d] & \mathcal{H}om(\mathcal{E}', \mathcal{N}(b)) \\
\mathcal{H}om(\mathcal{L}, \mathcal{G}(b)). &
\end{tikzcd}$$

We may assume that b is so large that $\mathcal{H}om(\mathcal{R},\mathcal{N}(b))$ is generated by global sections.

Let $W \subseteq H^0(\mathcal{H}om(\mathcal{L},\mathcal{G}(b))$ be the image of $H^0(\mathcal{H}om(\mathcal{R},\mathcal{N}(b))$ above. As before, since the sheaves $\mathcal{R}$ and $\mathcal{L}$ are locally free in codimension 1, the cokernels of these maps of sheaves are $\mathcal{E}xt$ sheaves with support in codimension ≥ 2, so W will satisfy the conditions of (3.2) and generate a subsheaf of rank 2. Therefore a general element $s \in W$ will give a map $\mathcal{L}(-b) \to \mathcal{G}$ whose quotient is torsion-free and locally free in codimension 1. However, instead of using this map, we add it to the existing map s_0 of $\mathcal{L}(-b)$ to $\mathcal{G}$, and the proof of (3.2) shows that for general $s \in W$, the sum $s_0 + s$ will give a good cokernel. Since by construction s comes from a map of $\mathcal{R}$ to $\mathcal{N}(b)$, we get maps of $\mathcal{E}(-b)$ to $\mathcal{N}$ and $\mathcal{E}'(-b)$ to $\mathcal{N}$, which we add to the existing maps. Thus we get a diagram like (2) above, but with new maps, where the cokernel of the bottom two rows is $\mathcal{I}_{V'}(a')$, with V' in the same biliaison class as V_2, for some $a' \in \mathbb{Z}$, and where the map $\mathcal{E}'(-b)$ to $\mathcal{N}$ extends to $\mathcal{E}(-b)$, as required.

Lemma 4.6. *Again with X normal* ACM*, suppose given V of codimension* 2 *and exact sequences*

$$0 \to \mathcal{E}_0 \oplus \mathcal{E}' \oplus \mathcal{L} \to \mathcal{N} \to \mathcal{I}_V(a) \to 0$$

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{L} \to 0$$

with $\mathcal{E}_0, \mathcal{N}$ coherent, $\mathcal{E}', \mathcal{E}$ layered ACM *sheaves and $\mathcal{L}$ a* rank 1 ACM *sheaf. Then for any $b \gg 0$ there is an exact sequence*

$$0 \to \mathcal{E}_0 \oplus \mathcal{E}(-b) \to \mathcal{N} \to \mathcal{I}_{V'}(a') \to 0$$

with V' in the same biliaison class as V.

Proof. By (4.5) we can find

$$0 \to \mathcal{E}_0 \oplus \mathcal{E}'(-b_1) \oplus \mathcal{L} \to \mathcal{N} \to \mathcal{I}_{V_1}(a_1) \to 0$$

where the map $\mathcal{E}'(-b_1)$ to $\mathcal{N}$ extends to a map of $\mathcal{E}(-b_1)$ to $\mathcal{N}$. Next, by the method of (3.4) we replace $\mathcal{L}$ by $\mathcal{L}(-b_2)$, and we may assume $b_1 = b_2 = b$ by taking both sufficiently large. Now the same proof as for the latter part of (4.5) shows how to get the desired sequence.

Proof of (4.3). Let V be a codimension 2 ACM subscheme of X. Take a set of generators for $H^0_*(\mathcal{I}_V)$ and thus obtain an $\mathcal{E}$-type resolution

$$0 \to \mathcal{E} \to \mathcal{L} \to \mathcal{I}_V \to 0$$

with $\mathcal{L}$ dissocié and $H^1_*(\mathcal{E}) = 0$. Since V is ACM, it follows that $\mathcal{E}$ is an ACM sheaf on X.

By (4.4), using condition (B) and hence (B*), we can find a resolution

$$0 \to \mathcal{E} \to \mathcal{G}_1 \to \mathcal{G}_2 \to 0$$

with $\mathcal{G}_1, \mathcal{G}_2$ layered ACM sheaves. Taking $\mathcal{H}$ to be the fibered sum of $\mathcal{L}$ and $\mathcal{G}_1$ over $\mathcal{E}$, we obtain sequences

$$0 \to \mathcal{G}_1 \to \mathcal{H} \to \mathcal{I}_V \to 0$$

and

$$0 \to \mathcal{L} \to \mathcal{H} \to \mathcal{G}_2 \to 0.$$

Thus $\mathcal{H}$ is layered, and we have a resolution of $\mathcal{I}_V$ with $\mathcal{G}_1, \mathcal{H}$ both layered ACM sheaves. (In fact, since $\mathcal{L}$ is dissocié and $\mathcal{G}_2$ is ACM, this last sequence splits, and $\mathcal{H} \cong \mathcal{L} \oplus \mathcal{G}_2$, but we do not need this.)

Let V' be another codimension 2 ACM subscheme of X, and let

$$0 \to \mathcal{G}_1' \to \mathcal{H}' \to \mathcal{I}_{V'} \to 0$$

be a similar sequence for V'. Now adding $\mathcal{H}'$ to the sequence for V and $\mathcal{H}$ to the sequence for V' we obtain resolutions

$$0 \to \mathcal{G}_1 \oplus \mathcal{H}' \to \mathcal{H} \oplus \mathcal{H}' \to \mathcal{I}_V \to 0$$

$$0 \to \mathcal{G}_1' \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}' \to \mathcal{I}_{V'} \to 0$$

by layered ACM sheaves, with the same sheaf in the middle of each. Under hypothesis a) we add $(\mathcal{H} \oplus \mathcal{H}')^\vee$ to both sequences, or under hypothesis b) we add a suitable sum $\oplus \mathcal{L}_i$ of rank 1 ACM sheaves to each, and then we obtain sequences (changing notation)

$$0 \to \mathcal{E} \to \mathcal{N} \to \mathcal{I}_V \to 0$$

$$0 \to \mathcal{E}' \to \mathcal{N} \to \mathcal{I}_{V'} \to 0$$

where $\mathcal{E}, \mathcal{E}'$ and $\mathcal{N}$ are all three orientable layered ACM sheaves on X.

Next, we invoke condition (C) to tell us that $\mathcal{E}$ and $\mathcal{E}'$ are both equal to $r \cdot \mathcal{O}$ in the group G', and hence equal to each other in G'. What does this equality mean? It means that up to twist, we can transform $\mathcal{E}$ to $\mathcal{E}'$ by a finite number of operations of adding and subtracting expressions $(\mathcal{F} - \mathcal{F}' - \mathcal{F}'')$ whenever $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence in the category $\mathcal{M}$ of layered ACM sheaves on X. In fact, it is enough to consider such expressions in which $\mathcal{F}''$ has rank 1, because any exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ of layered ACM sheaves can be resolved into a finite number of such sequences whose last term has rank 1, by using the filtration on $\mathcal{F}''$. By moving negative terms to the other side of the equation, we find a layered ACM sheaf $\mathcal{G}$ such that one can transform $\mathcal{E} \oplus \mathcal{G}$ into $\mathcal{E}' \oplus \mathcal{G}$ by a finite number of substitutions of $\mathcal{F}$ for $\mathcal{F}' \oplus \mathcal{F}''$ or vice versa, when $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence as above. All this takes place up to twists.

Now (3.4) allows us to replace a layered ACM sheaf by the direct sum of its factors (sufficiently twisted), and (4.6) allows us to replace a direct sum $\mathcal{E}' \oplus \mathcal{L}$ by an extension $\mathcal{E}$ of $\mathcal{L}$ by $\mathcal{E}'$, sufficiently twisted. By applying these lemmas to the resolution

$$0 \to \mathcal{E} \oplus \mathcal{G} \to \mathcal{N} \oplus \mathcal{G} \to \mathcal{I}_V \to 0$$

of V we thus obtain a resolution

$$0 \to \mathcal{E}'(-b) \oplus \mathcal{G}_1 \to \mathcal{N} \oplus \mathcal{G} \to \mathcal{I}_{V_1}(a_1) \to 0$$

for some V_1 in the biliaison class of V, and where $\mathcal{G}$ is a layered ACM sheaf, and $\mathcal{G}_1$ differs from $\mathcal{G}$ only by twists of its direct summands.

We compare this to the resolution

$$0 \to \mathcal{E}' \oplus \mathcal{G} \to \mathcal{N} \oplus \mathcal{G} \to \mathcal{I}_{V'} \to 0$$

of V'. Note that the sheaves $\mathcal{E}'(-b) \oplus \mathcal{G}_1$ and $\mathcal{E}' \oplus \mathcal{G}$ have all the same rank 1 factors up to twist. So from (3.1) we conclude that V, V_1 and V' are all in the same biliaison class.

Theorem 4.7. *Let X be a three-dimensional normal arithmetically Gorenstein projective scheme. Assume condition* (C) *and the stronger condition* (B$''$)*: every orientable* ACM *sheaf on X is layered. Then two locally Cohen–Macaulay curves C_1, C_2 in X are in the same Gorenstein biliaison class if and only if their Rao modules $H^1_*(\mathcal{I}_{C_1})$ and $H^1_*(\mathcal{I}_{C_2})$ are isomorphic, up to twist.*

Proof. One direction is elementary, since we do biliaison on ACM surfaces (1.2).

For the other direction, let C be any locally Cohen–Macaulay curve on X, with Rao module $M = H^1_*(\mathcal{I}_C)$. Take an $\mathcal{N}$-type resolution of C, that is, an exact sequence

$$0 \to \mathcal{L} \to \mathcal{N} \to \mathcal{I}_C \to 0$$

with $\mathcal{L}$ dissocié, and $\mathcal{N}$ a locally Cohen–Macaulay sheaf with $H^1_*(\mathcal{N}) = M$ and $H^2_*(\mathcal{N}) = 0$ [8, 1.12,1.13]. Let $N = H^0_*(\mathcal{N})$. As in [8, 3.2] a two-step resolution of N

$$0 \to P \to L_1 \to L_0 \to N \to 0$$

over the homogeneous coordinate ring S of X gives a maximal Cohen–Macaulay module P, and dualizing this sequence gives a sequence

$$0 \to N^\vee \to L_0^\vee \to L_1^\vee \to P^\vee \to M^* \to 0,$$

where $\cdot^\vee = \mathrm{Hom}_S(\cdot, S)$ and $M^* = \mathrm{Hom}_k(M, k)$.

Now take a minimal free resolution of M^*,

$$0 \to R \to F_2 \to F_1 \to F_0 \to M^* \to 0$$

defining a new S-module R. Let $\mathcal{N}'$ be the sheaf associated to $R^\vee$. There is a natural map of the free resolution of M^* into the earlier resolution, and so we get a map $R \to N^\vee$ giving rise to a map of sheaves $\mathcal{N} \to \mathcal{N}'$, which induces an isomorphism on the H^1_*-modules, both isomorphic to M. By adding suitable dissocié sheaves to the original $\mathcal{L}$ and $\mathcal{N}$, we may assume that $\mathcal{N} \to \mathcal{N}'$ is surjective also on H^0_*, so we get

$$0 \to \mathcal{E} \to \mathcal{N} \to \mathcal{N}' \to 0$$

with $\mathcal{E}$ an ACM sheaf on X.

Let $\mathcal{L}'$ be a dissocié sheaf mapping to $\mathcal{N}'$

$$0 \to \mathcal{L}' \to \mathcal{N}' \to \mathcal{I}_{C'}(a') \to 0$$

with cokernel $\mathcal{I}_{C'}(a')$ for some curve C'. Composing with the map $\mathcal{N} \to \mathcal{N}'$, we get

$$0 \to \mathcal{E}' \to \mathcal{N} \to \mathcal{I}_{C'}(a') \to 0$$

where $\mathcal{E}'$ is an extension of $\mathcal{L}'$ by $\mathcal{E}$ (actually a direct sum), hence another ACM sheaf.

Now by hypothesis (B$''$), $\mathcal{E}'$ is a layered ACM sheaf, and it is also orientable, since $\mathcal{N}$ is.

At this point we can repeat the last part of the proof of (4.3), using condition (C), and noting that for that part of the proof, it was not necessary for $\mathcal{N}$ to be ACM. Thus C and C' are in the same Gorenstein biliaison class. Since the biliaison class of C' depends only on the Rao module, and not on C, this shows that any two curves with the same Rao module are in the same biliaison class.

5 Applications I: Schemes of finite representation type

We say a projective ACM scheme X has *finite representation type* if there are only finitely many isomorphism classes of indecomposable ACM sheaves on X up to twist. The projective schemes of finite representation type have been classified by Eisenbud and Herzog, see [18, 17.10]. Those of dimension ≥ 2 are (i) $\mathbb{P}^n$ for $n \geq 2$, (ii) a non-singular quadric hypersurface Q^n in $\mathbb{P}^{n+1}$ for $n \geq 2$, (iii) the rational cubic scroll in $\mathbb{P}^4$, and (iv) the Veronese surface in $\mathbb{P}^5$.

If X is $\mathbb{P}^n$, our conditions (B) and (C) are trivial, because the only ACM sheaves on X are dissocié. Theorem (4.3) tells us that all codimension 2 ACM schemes are in the same Gorenstein biliaison class, and (4.7) tells us in $\mathbb{P}^3$ that the Gorenstein biliaison class of any curve is determined by its Rao module. Since Gorenstein biliaison in codimension 2 in $\mathbb{P}^n$ is just complete intersection biliaison, we recover the well-known results of Apéry–Gaeta–Peskine–Szpiro [16] and of Rao [17]. Our proof cannot be considered a new proof, because the methods we use are generalizations of the earlier proofs.

If X is a non-singular quadric hypersurface Q^n in $\mathbb{P}^{n+1}$, over an algebraically closed field k of characteristic $\neq 2$, the non-free indecomposable ACM sheaves have been classified by the work of Buchweitz, Eisenbud, and Herzog [1] and Knörrer [11], see [18, 14.10]. Up to isomorphism and twist there is just one non-free ACM sheaf if n is odd, or two if n is even. The rank of these sheaves is 2^m where $m = \left[\frac{n-1}{2}\right]$.

Proposition 5.1. *If X is a non-singular quadric surface in $\mathbb{P}^3$, all zero-dimensional subschemes of X are in the same Gorenstein biliaison class.*

Proof. In this case there are two non-trivial indecomposable ACM sheaves of rank 1, namely $\mathcal{I}_L$ and $\mathcal{I}_M$ where L, M represent the two classes of lines on X. Thus any ACM sheaf is a direct sum of rank 1 ACM sheaves, and condition (B) is trivial.

For condition (C), note that $L + M = H$, the hyperplane class, and since two lines in the same family do not meet, there is an exact sequence

$$0 \to \mathcal{I}_L \to \mathcal{O}^2 \to \mathcal{I}_M(1) \to 0.$$

If $\mathcal{E}$ is an orientable ACM sheaf, it must have the same number of L's and M's in its direct sum decomposition, so the existence of this sequence proves (C).

We conclude the result from (4.3).

Remark 5.2. This generalizes an earlier result [9, 2.3] which showed that any set of n general points on the quadric surface was in the biliaison class of a point.

Example 5.3. If X is a non-singular quadric hypersurface of dimension $n \geq 3$ in $\mathbb{P}^{n+1}$, then Pic $X = \mathbb{Z}$, generated by $\mathcal{O}(1)$, and there are no non-trivial extensions, so condition (C) holds trivially. On the other hand, there are indecomposable ACM sheaves of rank ≥ 2. Any sequence as in condition (B) would have to split, so condition (B) fails. We conclude there are many inequivalent Gorenstein biliaison classes of codimension 2 subschemes of X. In fact, since Pic $X = \mathbb{Z}$, Gorenstein biliaison is the same as CI-biliaison, and the biliaison classes are determined by stable equivalence classes (up to twist) of ACM sheaves on X [8, 2.4].

Proposition 5.4. *All zero-dimensional subschemes of the rational cubic scroll X in $\mathbb{P}^4$ are in the same Gorenstein biliaison class.*

Proof. First recall that Pic $X \cong \mathbb{Z}^2$ and is generated by the hyperplane class H and the class of a fiber F of the ruled surface. It is well known [15, 5.10] that up to twist the only rank 1 ACM sheaves are $\mathcal{O}_X$, $\mathcal{I}_F$, $\mathcal{I}_{H-F}$, $\mathcal{I}_{H-2F}$. Here $H - F$ is a conic, and $H - 2F$ is the exceptional line E with $E^2 = -1$.

Looking at the proof [18, 16.12] that X is of finite representation type, we find that there is just one (up to twist) indecomposable ACM sheaf of rank > 1, it has rank 2, and is obtained as the first syzygy of $\mathcal{I}_F$:

$$0 \to \mathcal{E}_0 \to \mathcal{O}(-1)^3 \to \mathcal{I}_F \to 0.$$

Note that $c_1(\mathcal{E}_0) = F - 3H$.

First we will show that $\mathcal{E}_0$ is isomorphic to an extension of rank 1 ACM sheaves

$$0 \to \mathcal{I}_F(-1) \to \mathcal{F} \to \mathcal{I}_{H-2F}(-1) \to 0. \tag{1}$$

We compute $\operatorname{Ext}^2(\mathcal{I}_{H-2F}(-1), \mathcal{I}_F(-1)) = H^1(\mathcal{I}_{3F}(1))$ is of dimension 1, so there exists a non-trivial extension (1) and it is an ACM sheaf of rank 2. Next, note that $h^0(\mathcal{F}(1)) = 0$, $h^0(\mathcal{F}(2)) = 6$. So if $\mathcal{F}$ were decomposable, then $\mathcal{F} = \mathcal{L}_1 \oplus \mathcal{L}_2$, where $\mathcal{L}_1, \mathcal{L}_2$ are rank 1 ACM sheaves with $h^0(\mathcal{L}_i(1)) = 0$, and at least one of $h^0(\mathcal{L}_i(2)) \neq 0$. The only rank 1 ACM sheaves $\mathcal{L}$ with $h^0(\mathcal{L}(1)) = 0$, $h^0(\mathcal{L}(2)) \neq 0$ are $\mathcal{O}(-2)$, $\mathcal{I}_F(-1)$, $\mathcal{I}_{H-F}(-1)$, $\mathcal{I}_{H-2F}(-1)$. For these sheaves $\mathcal{L}$ we have $h^0(\mathcal{L}(2)) = 1, 3, 2, 3$, respectively. Taking into account $c_1(\mathcal{F}) = F - 3H$, the only possibility would be $\mathcal{F} \cong \mathcal{I}_F(-1) \oplus \mathcal{I}_{H-2F}(-1)$, which is impossible since the sequence (1) is non-split.

Therefore $\mathcal{F}$ is an indecomposable rank 2 ACM sheaf on X. Since there is only one such up to twist [18, 16.12], we find checking Chern classes that $\mathcal{F} \cong \mathcal{E}_0$. Thus $\mathcal{E}_0$ is already layered, and condition (B) of (4.3) is satisfied.

Note that X is not arithmetically Gorenstein, but that the rank 1 ACM divisors H, F, $H - F$ and their twists generate $\operatorname{Pic} X$ as a monoid, so condition b) of (4.3) is satisfied.

For condition (C), note that since $F \cdot F = 0$, there is an exact sequence

$$0 \to \mathcal{I}_F \to \mathcal{O}^2 \to \mathcal{I}_{H-F}(1) \to 0, \tag{2}$$

and since $F \cdot 2F = 0$ another sequence

$$0 \to \mathcal{I}_F \to \mathcal{O}_X \oplus \mathcal{I}_{H-F}(1) \to \mathcal{I}_{H-2F}(1) \to 0. \tag{3}$$

Now suppose given an orientable ACM sheaf on X. Up to twists we can write it as a direct sum

$$a \cdot \mathcal{I}_F + b \cdot \mathcal{I}_{H-F} + c\mathcal{I}_{H-2F} + d \cdot \mathcal{E}_0 + e \cdot \mathcal{O}$$

with $a, b, c, d, e \geq 0$.

In the quotient Grothendieck group G', we use sequence (1) to replace $\mathcal{E}_0$ by $\mathcal{I}_F + \mathcal{I}_{H-2F}$ and then we may assume $d = 0$. Since the sheaf is orientable, if $c \neq 0$, then also $a \neq 0$. So we can use sequence (3) to replace occurrences of $\mathcal{I}_{H-2F} + \mathcal{I}_F$ by $\mathcal{I}_{H-F} + \mathcal{O}$ and then assume $c = 0$. Now if b is non-zero, it must be equal to a, and we use sequence (2) to reduce to a multiple of $\mathcal{O}$. Thus condition (C) is satisfied.

Thus by (4.3) every zero-scheme on X is in the biliaison class of a point.

Remark 5.5. This generalizes an earlier result [3, 3.4] that treated only sets of points in general position.

Proposition 5.6. *On the Veronese surface X in $\mathbb{P}^5$ there are infinitely many biliaison classes of zero-schemes, indexed by the even integers.*

Proof. The Veronese surface is the 2-tuple embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$. Thus $\operatorname{Pic} X = \mathbb{Z}$, generated by the image of a line of $\mathbb{P}^2$, which becomes a conic $C \subseteq X$. The hyperplane class is $H = 2C$. Every curve on X is an ACM curve. X is not arithmetically Gorenstein, but its Picard group is generated as a monoid by C and $H - C$, which are ACM curves. There are no non-trivial extensions of rank 1 ACM sheaves, so property (C) fails, since $\mathcal{I}_C \oplus \mathcal{I}_C$ is an orientable ACM sheaf not equivalent in the group G' to $2 \cdot \mathcal{O}$.

Following the proof of [18, 16.10] we see that there is just one (up to twist) indecomposable ACM sheaf of rank > 1; it has rank 2 and is obtained as the first syzygy of $\mathcal{I}_C$:

$$0 \to \mathcal{E}_0 \to \mathcal{O}(-1)^3 \to \mathcal{I}_C \to 0.$$

Pulling this sequence back to $\mathbb{P}^2$ one recognizes that $\mathcal{E}_0 \cong \Omega^1_{\mathbb{P}^2}(-1)$. Note that $H^1(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}) \neq 0$, but still $\mathcal{E}_0$ is an ACM sheaf on X because twists on X correspond to even twists on $\mathbb{P}^2$, and no other twist of $\Omega^1_{\mathbb{P}^2}$ has a non-zero H^1. Now $\mathcal{E}_0(2)$ is the tangent bundle on X, and transporting a well-known sequence with the tangent bundle from $\mathbb{P}^2$ we find

$$0 \to \mathcal{O}(-2) \to \mathcal{I}_C(-1)^3 \to \mathcal{E}_0 \to 0,$$

so condition (B) is satisfied.

Since condition (C) fails, (4.2) tells us that there is more than one biliaison class of zero-schemes on X. This is clear anyway, because $H.Y$ is even for any curve Y on X, so biliaison preserves the parity of the length of a zero-scheme Z.

To investigate more exactly what the biliaison classes are, let Z, Z' be any two zero-schemes in X. Since condition (B) holds, we can use (4.4) and the beginning of the proof of (4.3) to find resolutions

$$0 \to \mathcal{E} \to \mathcal{N} \to \mathcal{I}_Z \to 0$$

$$0 \to \mathcal{E}' \to \mathcal{N}' \to \mathcal{I}_{Z'} \to 0$$

where $\mathcal{E}, \mathcal{E}'$ $\mathcal{N}, \mathcal{N}'$ are orientable layered ACM sheaves, and we may even assume $\mathcal{N} = \mathcal{N}'$. In our case, this means $\mathcal{E}, \mathcal{E}', \mathcal{N}, \mathcal{N}'$ are each direct sums of copies of $\mathcal{O}(a_i)$ and $\mathcal{I}_C(b_i)$ for various a_i, b_i.

Without assuming $\mathcal{N} = \mathcal{N}'$, if Z and Z' are in the same biliaison class, then the exact sequences of (4.1) show that there are layered ACM sheaves $\mathcal{F}, \mathcal{F}'$ with the same rank 1 factors, up to twist, and an equivalence $\mathcal{N} + \mathcal{E}' + \mathcal{F} = \mathcal{N}' + \mathcal{E} + \mathcal{F}'$ in the Grothendieck group G'. Since $\mathcal{F}$ and $\mathcal{F}'$ have the same factors, up to twist, we find $\mathcal{N} + \mathcal{E}' = \mathcal{N}' + \mathcal{E}$ in G'.

Let $m(Z) =$ the number of copies of $\mathcal{I}_C$ in the direct sum decomposition of $\mathcal{N}$, minus the number in $\mathcal{E}$. Since $\mathcal{E}, \mathcal{N}$ are orientable, $m(Z)$ is an even integer. What (4.1) tells us in this case is that $m(Z)$ is an invariant of the biliaison class.

Conversely, if Z, Z' are two schemes with $m(Z) = m(Z')$, then as in the proof of (4.3), we find the sequences above with $\mathcal{N} = \mathcal{N}'$. Then $\mathcal{E}$ and $\mathcal{E}'$ have the same number of direct summands $\mathcal{I}_C(a_i)$ for various a_i, and it follows from (3.1) that they are in the same biliaison class.

Given $m > 0$, m even, take $\mathcal{N} = \oplus_{i=1}^m \mathcal{I}_C$, choose a dissocié sheaf $\mathcal{L}$ such that

$$0 \to \mathcal{L} \to \mathcal{N} \to \mathcal{I}_Z(a) \to 0$$

for some zero-scheme Z, and then Z will have invariant m.

If $m < 0$, m even, let $n = -m$. Take a general $n \times (n+1)$ matrix of linear forms on $\mathbb{P}^2$, and let Z be the associated determinantal scheme, so that we have a resolution

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-n-1)^n \to \mathcal{O}_{\mathbb{P}^2}(-n)^{n+1} \to \mathcal{I}_{Z,\mathbb{P}^2} \to 0.$$

Transporting this to the Veronese surface gives Z with resolution

$$0 \to \mathcal{I}_C\left(-\frac{n}{2}\right)^n \to \mathcal{O}\left(-\frac{n}{2}\right)^{n+1} \to \mathcal{I}_Z \to 0$$

on the Veronese surface, showing that Z has invariant $-n = m$.

Thus the biliaison classes of zero-schemes on X are in one-to-one correspondence with the even integers via the invariant $m(Z)$.

Example 5.7. Let X be the Veronese surface in $\mathbb{P}^5$. If $Z = P$ is a single point, there is a resolution

$$0 \to \mathcal{O}(-1) \to \mathcal{I}_C \oplus \mathcal{I}_C \to \mathcal{I}_P \to 0,$$

since two lines in $\mathbb{P}^2$ meet in a point. Thus $m(P) = 2$. It is easy to see that $Z =$ three points in general position has $m(Z) = -2$. On the other hand, the image in X of 3 points on a line in $\mathbb{P}^2$ is in the biliaison class of a point, so $m = 2$.

If Z is two points, there is a resolution

$$0 \to \mathcal{I}_C(-1) \to \mathcal{O}(-1) \oplus \mathcal{I}_C \to \mathcal{I}_Z \to 0$$

since a line and a conic in $\mathbb{P}^2$ meet in 2 points. Hence $m(Z) = 0$, and we conclude that 2 points are in the biliaison class of a complete intersection in X. This is not obvious! We leave as an amusing exercise for the reader to find explicit biliaisons (among non-empty zero schemes) that relate 2 points to a complete intersection in X.

To find a scheme with $m(Z) = 4$, we use a sequence

$$0 \to \mathcal{O}(-1)^3 \to \mathcal{I}_C^4 \to \mathcal{I}_Z(1) \to 0.$$

Then Z is a set of 6 points, which may be taken in general position.

6 Applications, II: Quadric cones

In this section we consider some schemes that are not of finite representation type, but on which we can still determine all possible ACM sheaves. We use the matrix factorizations of Eisenbud [5], see [18, Ch. 7] and the periodicity theorems of Knörrer [11], see [18, Ch. 12].

Proposition 6.1. *Let X be a quadric cone in $\mathbb{P}^3$. Then all zero-schemes on X are in the same Gorenstein biliaison class.*

Proof. Recall that APic $X = \mathbb{Z}$, generated by a line L, and $2L = H$ is the hyperplane class. The surface is arithmetically Gorenstein, and there is an exact sequence

$$0 \to \mathcal{I}_L \to \mathcal{O}^2 \to \mathcal{I}_L(1) \to 0,$$

so condition (C) is satisfied.

To determine ACM sheaves of higher rank, we use a result of Buchweitz, Greuel, and Schreyer [2, 4.1]. If $S = k[x,t]$ and $R = S/(x^2)$, they found that there are three types of indecomposable maximal Cohen–Macaulay modules on R, given by matrix factorizations (φ, ψ)

(i) $((x^2),(1))$

(ii) $((x),(x))$

(iii) $(\varphi_\ell, \psi_\ell)$ where $\varphi_\ell = \psi_\ell = \begin{pmatrix} x & t^\ell \\ 0 & -x \end{pmatrix}$ for $\ell = 1,2,3,\ldots$.

The corresponding R-module will be the cokernel of a map of free S-modules determined by the first matrix in each pair. Thus (i) gives $S/(x^2) = R$, (ii) gives $S/(x) = R/(x)$, and (iii) gives a module isomorphic to the ideal (x, t^ℓ) in R.

By Knörrer's periodicity theorem [11, 3.1], if we let $S_2 = k[x,u,v,t]$ and $R_2 = S_2/(x^2 + uv)$, then the MCM modules on R' are given by matrix factorizations

$$\left(\begin{pmatrix} u & \psi \\ \varphi & -v \end{pmatrix}, \begin{pmatrix} v & \psi \\ \varphi & -u \end{pmatrix} \right)$$

for each matrix factorization (φ, ψ) of an MCM module on R.

Under this correspondence, type (i) gives us R_2; type (ii) gives us the ideal $I_L = (x,u)$ of a line in X, and type (iii) gives a module M_ℓ, for $\ell = 1,2,3,\ldots$, given by a matrix factorization

$$\left(\begin{pmatrix} u & 0 & x & t^\ell \\ 0 & u & 0 & -x \\ x & t^\ell & -v & 0 \\ 0 & -x & 0 & -v \end{pmatrix}, \begin{pmatrix} v & 0 & x & t^\ell \\ 0 & v & 0 & -x \\ x & t^\ell & -u & 0 \\ 0 & -x & 0 & -u \end{pmatrix} \right).$$

This is equivalent, permuting columns and rows to

$$\left(\begin{pmatrix} u & x & 0 & t^\ell \\ x & -v & t^\ell & 0 \\ 0 & 0 & u & -x \\ 0 & 0 & -x & -v \end{pmatrix}, \begin{pmatrix} v & x & 0 & t^\ell \\ x & -u & t^\ell & 0 \\ 0 & 0 & v & -x \\ 0 & 0 & -x & u \end{pmatrix}\right) .$$

Thus we see that M_ℓ is an extension

$$0 \to I_L \to M_\ell \to I_L(-\ell+1) \to 0$$

of rank 1 R_2-modules. The corresponding ACM sheaf $\mathcal{E}_\ell$ on X is therefore an extension

$$0 \to \mathcal{I}_L \to \mathcal{E}_\ell \to \mathcal{I}_L(-\ell+1) \to 0$$

of rank 1 ACM sheaves on X.

Thus every ACM sheaf on X is layered, and condition (B) is satisfied.

Now the result follows from (4.3).

Theorem 6.2. *Let X be the singular quadric three-fold in $\mathbb{P}^4$ that is the cone over a non-singular quadric surface in $\mathbb{P}^3$. Two locally Cohen–Macaulay curves on X are in the same Gorenstein biliaison class if and only if their Rao modules are isomorphic. In particular, all* ACM *curves are in the same biliaison class.*

Proof. We will show that conditions (B), (C) of (4.3) and (B$''$) of (4.7) hold, and conclude by applying those two theorems.

Again we use the result of [2, 4.1] mentioned in the previous proof, but in this case we will have to use Knörrer's double branched covers theorem [11, 2.5]: if $S_1 = k[x, y, t]$ and $R_1 = S_1/(x^2 + y^2)$, then every indecomposable MCM module on R_1 is a direct summand of a module with matrix factorization

$$\left(\begin{pmatrix} y & \psi \\ \varphi & -y \end{pmatrix}, \begin{pmatrix} y & \psi \\ \varphi & -y \end{pmatrix}\right)$$

for each matrix factorization (φ, ψ) of an indecomposable MCM module over R.

From (i) we obtain the ring R_1. From (ii) we obtain a matrix factorization

$$\left(\begin{pmatrix} y & x \\ x & -y \end{pmatrix}, \begin{pmatrix} y & x \\ x & -y \end{pmatrix}\right) .$$

Setting $a = x + iy$, $b = x - iy$, we see this matrix factorization is equivalent to

$$\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}\right) .$$

The corresponding module is a direct sum of $S_1/(a)$ and $S_1/(b)$.

From type (iii) we obtain a matrix factorization

$$\left(\begin{pmatrix} y & 0 & x & t^\ell \\ 0 & y & 0 & -x \\ x & t^\ell & -y & 0 \\ 0 & -x & 0 & -y \end{pmatrix}, \text{ ditto}\right).$$

Again setting $x^2 + y^2 = ab$, we find this matrix factorization is equivalent to one that corresponds to a direct sum of two modules M_ℓ and M'_ℓ with matrix factorizations

$$\left(\begin{pmatrix} a & -t^\ell \\ 0 & b \end{pmatrix}, \begin{pmatrix} b & t^\ell \\ 0 & a \end{pmatrix}\right)$$

and

$$\left(\begin{pmatrix} b & -t^\ell \\ 0 & a \end{pmatrix}, \begin{pmatrix} a & t^\ell \\ 0 & b \end{pmatrix}\right).$$

Thus we have shown that indecomposable MCM modules over R_1 correspond to matrix factorizations of 5 types

(a) $((x^2 + y^2), (1))$

(b) $((a), (b))$

(c) $((b), (a))$

(d) $\left(\begin{pmatrix} a & -t^\ell \\ 0 & b \end{pmatrix}, \begin{pmatrix} b & t^\ell \\ 0 & a \end{pmatrix}\right)$, $\ell = 1, 2, \ldots$

(e) $\left(\begin{pmatrix} b & -t^\ell \\ 0 & a \end{pmatrix}, \begin{pmatrix} a & t^\ell \\ 0 & b \end{pmatrix}\right)$, $\ell = 1, 2, \ldots$.

Now we use Knörrer's periodicity theorem [11, 3.1] to pass to the rings $S_3 = k[x, y, u, v, t]$ and $R_3 = S_3/(x^2 + y^2 + uv)$. Preserving the notation $ab = x^2 + y^2$, as in the proof of (6.1) to each of these matrix factorizations (φ, ψ) we obtain matrix factorizations

$$\left(\begin{pmatrix} u & \psi \\ \varphi & -v \end{pmatrix}, \begin{pmatrix} v & \psi \\ \varphi & -u \end{pmatrix}\right)$$

over the ring S_3.

From case (a) we obtain the ring R_3. From cases (b) and (c) we obtain modules isomorphic to the ideals (a, u) and (a, v) in R_3. These correspond to the two families of planes D, E passing through the vertex of the cone X.

From case (d) we obtain a matrix factorization

$$\left(\begin{pmatrix} u & 0 & a & -t^\ell \\ 0 & u & 0 & b \\ b & t^\ell & -v & 0 \\ 0 & a & 0 & -v \end{pmatrix}, \begin{pmatrix} v & 0 & a & -t^\ell \\ 0 & v & 0 & b \\ b & t^\ell & -u & 0 \\ 0 & a & 0 & -u \end{pmatrix}\right).$$

As in the proof of (6.1), we rearrange rows and columns, and then observe that the corresponding module M_ℓ is an extension

$$0 \to I_D \to M_\ell \to I_E(-\ell+1) \to 0.$$

Similarly in case (e) we obtain a module M'_ℓ that is an extension

$$0 \to I_E \to M'_\ell \to I_D(-\ell+1) \to 0.$$

Translating this in terms of the ACM sheaves on X, we find that the indecomposable ACM sheaves on X, up to twist, are $\mathcal{O}_X$, $\mathcal{I}_D$, $\mathcal{I}_E$, and two infinite sequences $\mathcal{E}_\ell, \mathcal{E}'_\ell$ for $\ell = 1, 2, \ldots$ that are extensions

$$0 \to \mathcal{I}_D \to \mathcal{E}_\ell \to \mathcal{I}_E(-\ell+1) \to 0$$

$$0 \to \mathcal{I}_E \to \mathcal{E}'_\ell \to \mathcal{I}_D(-\ell+1) \to 0.$$

Thus we see that every ACM sheaf on X is layered, so X satisfies (B) and (B$''$). We know that APic $X = \mathbb{Z}^2$, generated by D, E and that $D + E = H.X$ is arithmetically Gorenstein, and there is an exact sequence

$$0 \to \mathcal{I}_D \to \mathcal{O}_X^2 \to \mathcal{I}_E(1) \to 0$$

so condition (C) is satisfied.

Now the result follows from (4.3) and (4.7).

Remark 6.3. Here is another way to construct the sheaves $\mathcal{E}_\ell$ and $\mathcal{E}'_\ell$. Let C be a plane curve of degree ℓ in the plane D. Then C is arithmetically Gorenstein with $\omega_C \cong \mathcal{O}_C(\ell-3)$. So by the Serre construction there is an extension

$$0 \to \mathcal{O}_X(-\ell) \to \mathcal{E} \to \mathcal{I}_C \to 0,$$

and $\mathcal{E}$ will be a rank 2 ACM sheaf. Since $C \subseteq D$, there is an inclusion $0 \to \mathcal{I}_D \to \mathcal{E}$, and chasing the corresponding diagram one finds the cokernel is $\mathcal{I}_E(-\ell+1)$. Thus $\mathcal{E}$ is the sheaf $\mathcal{E}_\ell$ described above.

Example 6.4. As an application of (6.2) we give a new proof and strengthening of a theorem of Lesperance. Let D_1, D_2 be two planes in the family of planes D, meeting at the singular point $P = (0,0,0,0,1)$ of X. Let $C_1 \subseteq D_1$ and $C_2 \subseteq D_2$ be curves of degrees d, t respectively, not containing the point P, with $2 \leq d \leq t$. Lesperance calls the union $C = C_1 \cup C_2$ a curve of type (P, d, t), and shows that its Rao module is $M_d = S_3/(x, y, u, v, t^d)$. Then he shows that two curves C and C' of types (P, d, t) and (P, d, s) with $2 \leq d \leq t, s$ are in the same G-liaison class [12, 4.9], [13, 3.6]. Since they have the same Rao module, we obtain a new proof of this result from (6.2), which shows they are also in the same G-biliaison class.

Next, Lesperance considers a curve $C' = C_1 \cup C_2'$ as above, where $C_1 \subseteq D_1$ does not contain P, but where $C_2' \subseteq D_2$ does contain P as a point of multiplicity e. This time we assume $d = \deg D_1 \geq 2$, but $t \geq e \geq 1$ only. This he calls a curve of type (P, d, t, e), and shows it has the same Rao module M_d. He shows that if $t \geq d + e$, then curves of type (P, d, t) and (P, d, t, e) are in the same even G-liaison class [12, 4.15], but he leaves open the question if $d = t$, and asks, for example if curves of types $(P, 2, 2)$ and $(P, 2, 2, 1)$ are G-linked. Again, since they have the same Rao modules, our theorem (6.2) shows that any curves of types (P, d, t) and (P, d, t, e) are in the same G-biliaison class. This answers the question [12, 4.14].

Note that our G-biliaisons take place inside a fixed singular quadric hypersurface X. But since any three skew lines in $\mathbb{P}^3$ are contained in a non-singular quadric surface, it is easy to see that if D_1', D_2' are any other two planes in $\mathbb{P}^4$ meeting at the same point P, then we can make G-biliaisons on different singular quadric hypersurfaces to relate curves of types (P, d, t) or (P, d, t, e) in D_1 and D_2 to those in D_1' and D_2'.

Example 6.5. Lesperance also gives an example [12, 5.7], [13, 4.6] of minimal curves in the same even G-liaison class, having the same degree and genus, that do not belong to the same irreducible family. His example consists of a curve C of type $(P, 2, 2, 1)$ on the one hand, and the disjoint union D of a line L and a twisted cubic curve Y, where L meets the $\mathbb{P}^3$ containing Y at the point P on the other hand. Both have degree 4, $p_a = -1$, and Rao module M_2.

We would like to point out that we can put curves of type C and D on a singular quadric 3-fold X, so that they are in the same G-biliaison class on X. This will show that minimal curves in a G-biliaison class on X, of the same degree and genus, need not form an irreducible family.

To do this, let Q be a non-singular quadric surface in $\mathbb{P}^3$. Take a point $O \in \mathbb{P}^4 \backslash \mathbb{P}^3$, and let X be the cone over Q with vertex O. Now take $Y \subseteq Q$ a twisted cubic curve, take $P \in Q \backslash Y$, and let L be the line joining O to P. Then $D = L \cup Y$ has Rao module M_2 based at P. Take a plane Λ_1 in $\mathbb{P}^3$ cutting Q in a conic C_1 containing P. Let M be a line in Q, passing through P, not in the plane Λ_1, and let Λ_2 be the plane spanned by M and O. Let $C_2 \subseteq \Lambda_2$ be a conic not passing through P. Then $C = C_1 \cup C_2$ is a curve of type $(P, 2, 2, 1)$. Since C and D have the same Rao module, they are in the same G-biliaison class on X, by

(6.2).

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