

Numerical computation of the asymptotic size of the rotation domain for the Arnold family

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ABSTRACT

We consider the Arnold Tongue of the Arnold family of circle maps associated to a fixed Diophantine rotation number θ . The corresponding maps of the family are analytically conjugate to a rigid rotation. This conjugation is defined on a (maximal) complex strip of the circle and, after a suitable scaling, the size of this strip is given by an analytic function of the perturbative parameter.

The main purpose of this paper is to perform a numerical accurate computation of this function and of its Taylor expansion. This allows us to verify previous theoretical results. The rotation numbers we select are quadratic irrationals, mainly the Golden Mean.

By introducing a nonstandard extrapolation process, especially suited for the problem, we compute all the quantities required (rotation numbers, Arnold Tongues, Fourier and Taylor coefficients) with high precision.

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1. Introduction

In this paper we consider the widely studied *Arnold family* of circle maps,

$$\begin{aligned} \tilde{f}_{\alpha,\varepsilon} : \mathbb{T}^1 &\longrightarrow \mathbb{T}^1 \\ x &\longrightarrow x + \frac{\alpha}{2\pi} + \frac{\varepsilon}{2\pi} \sin(2\pi x) \end{aligned} \quad (1)$$

where $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ and (α, ε) are real parameters. For any $\alpha \in [0, 2\pi)$ and $\varepsilon \in [0, 1)$, the map $\tilde{f}_{\alpha,\varepsilon}$ is an orientation-preserving analytic diffeomorphism of the circle and we denote by $\rho(\alpha, \varepsilon)$ its rotation number.

A well-known result on circle maps [1,8,10] ensures that, given f an analytic diffeomorphism of \mathbb{T}^1 , whose rotation number $\theta = \rho(f)$ is Diophantine, the map f is analytically conjugate to the rigid rotation $\mathcal{T}_\theta(x) = x + \theta$. Concretely, there exists an analytic diffeomorphism $\eta : \mathbb{T}^1 \rightarrow \mathbb{T}^1$ such that $\eta \circ \mathcal{T}_\theta = f \circ \eta$. If we require $\eta(0) = x_0$, for a fixed $x_0 \in \mathbb{T}^1$, then the conjugacy is unique. This conjugation can be written as

$$\eta(x) = x + \xi(x), \quad (2)$$

where ξ is a 1-periodic function. As η is (real) analytic, it can be analytically extended to a maximal complex strip of the form

$$\mathcal{A}(\Delta) = \{x \in \mathbb{C} : |\text{Im}(x)| < \Delta\}, \quad (3)$$

for some $\Delta > 0$. Abusing notation, we also denote by η this analytic extension. By the principle of analytic continuation, the map η still conjugates f to \mathcal{T}_θ in $\mathcal{A}(\Delta)$.

To apply this result to the Arnold family, we have to take into account the parametric dependence. Thus, for any $\theta \in [0, 1)$, the set $T_\theta = \{(\alpha, \varepsilon) : \rho(\alpha, \varepsilon) = \theta\}$ is called the *Arnold Tongue* of $\tilde{f}_{\alpha,\varepsilon}$ of rotation number θ . If θ is a Diophantine number, then T_θ is an analytic curve which is the graph of a function $\varepsilon \in [0, 1) \mapsto \alpha(\varepsilon)$, with $\alpha(0) = 2\pi\theta$ (see [16]). Hence, if we keep the Diophantine number θ fixed from now on, we have that the 1-parameter family of maps $\tilde{f}_{\alpha(\varepsilon),\varepsilon}$ is analytically conjugate to \mathcal{T}_θ through a family of analytic conjugations,

$$\tilde{\eta}_\varepsilon : \mathcal{A}(\tilde{\Delta}(\varepsilon)) \rightarrow \mathbb{C}, \quad (4)$$

also depending analytically on ε . Here, $\mathcal{A}(\tilde{\Delta}(\varepsilon))$ denotes the maximal strip in which $\tilde{\eta}_\varepsilon$ is defined.

For $\tilde{\Delta}(\varepsilon)$ we easily have that $\lim_{\varepsilon \rightarrow 1^-} \tilde{\Delta}(\varepsilon) = 0$ and $\lim_{\varepsilon \rightarrow 0^+} \tilde{\Delta}(\varepsilon) = +\infty$. In this paper we focus on the asymptotic behavior of $\tilde{\Delta}(\varepsilon)$ when $\varepsilon \rightarrow 0^+$. This problem was first considered in [4], where an asymptotic expression for this function was given.

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Concretely, if we write

$$\tilde{\Delta}(\varepsilon) = \frac{1}{2\pi} \log \tilde{R}_\varepsilon, \tag{5}$$

it was proved that

$$\tilde{R}_\varepsilon = \frac{2}{\varepsilon} R_\varepsilon = \frac{2}{\varepsilon} (R_0 + O(\varepsilon \log \varepsilon)). \tag{6}$$

Here, R_0 is the conformal radius of the Siegel disk at the origin of the so-called *complex semistandard map*

$$G(z) = ze^{i\omega} e^z, \tag{7}$$

where $\omega = 2\pi\theta$. Indeed, there exists a unique analytic diffeomorphism

$$\varphi : \mathbb{D}_{R_0} \rightarrow \mathbb{C} \tag{8}$$

such that $\varphi(0) = 0$, $\varphi'(0) = 1$ and $\varphi \circ \mathcal{R}_\omega = G \circ \varphi$, where $\mathbb{D}_{R_0} = \{z \in \mathbb{C} : |z| < R_0\}$ and $\mathcal{R}_\omega(z) = e^{i\omega} z$.

The estimate (6) was later improved in [2], where the authors proved that R_ε is an even analytic function in the unit disk \mathbb{D}_1 , so that

$$\tilde{R}_\varepsilon = \frac{2}{\varepsilon} (R_0 + O(\varepsilon^2)). \tag{9}$$

It is not difficult to give a geometrical view of this result. Let us consider an analytic map $F : \mathcal{U} \subset \mathbb{C} \rightarrow \mathbb{C}$ leaving the unit circle \mathbf{C}_1 invariant. We say that $F|_{\mathbf{C}_1}$ is *analytically linearizable* if there exists an analytic diffeomorphism $\varphi : \mathbf{C}_1 \rightarrow \mathbf{C}_1$, such that $\varphi \circ \mathcal{R}_\omega = F \circ \varphi$. If we ask $\varphi(1) = z_0$, for some $z_0 \in \mathbf{C}_1$, then φ is univocally defined. Being an analytic function on \mathbf{C}_1 , φ can be analytically continued to a maximal annulus around \mathbf{C}_1 of the form

$$A(R) = \{z \in \mathbb{C} : 1/R < |z| < R\}, \tag{10}$$

for some $R > 1$. Now, we consider $f : \mathbb{T}^1 \rightarrow \mathbb{T}^1$ the (analytic) circle map induced by $F|_{\mathbf{C}_1}$, using the exponential map $z = e^{2\pi i x}$, and we define $\eta : \mathbb{T}^1 \rightarrow \mathbb{T}^1$ so that

$$\varphi(e^{2\pi i x}) = e^{2\pi i \eta(x)}, \quad x \in \mathbb{T}^1, \tag{11}$$

with the normalization $\eta(0) = x_0$, where $e^{2\pi i x_0} = z_0$. Then, we have that $\eta \circ \mathcal{T}_\theta = f \circ \eta$, and thus, f is analytically conjugate to a rotation. Moreover, η is also analytic and the width of its strip of analyticity around \mathbb{T}^1 is $\Delta = (1/2\pi) \log R$. The image by φ of the maximal annulus $A(R)$ where φ can be analytically continued is called the *Herman ring* of F and the quantity $\Delta = (1/\pi) \log R$ is called the modulus of the ring.

Remark 1. We use the term *rotation domain* to refer to the image of the maximal domain of definition of an analytic conjugation to a rigid rotation of a circle map. We extend the term to refer to a Siegel disk or a Herman ring of an analytic map of \mathbb{C} , when there is no danger of confusion. For a Herman ring we call the R in Eq. (10), that defines the maximal annulus where the conjugation is defined, for the *size of the ring*. Similarly, for a Siegel disk we use its conformal radius R_0 of (8) to denote the size of the disk and, for a circle map, we measure the size of its rotation domain in terms of the width Δ of the strip of analyticity of the conjugation (3).

Now, we consider the *complex standard family*

$$\tilde{F}_{\alpha,\varepsilon}(z) = ze^{i\alpha} e^{\frac{\varepsilon}{2}(z-\frac{1}{z})}, \quad \alpha \in [0, 2\pi), \varepsilon \in [0, 1]. \tag{12}$$

This is a family of holomorphic maps of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ leaving \mathbf{C}_1 invariant. In \mathbb{T}^1 , this family of maps induces the Arnold family (1). Thus, the geometrical meaning of formula (9) is that, by means of a suitable scaling, the complex standard family becomes the semistandard map (7) when $(\alpha, \varepsilon) \rightarrow (\omega, 0)$ over T_θ , and the Herman ring of $\tilde{F}_{\alpha(\varepsilon),\varepsilon}$ becomes the Siegel disk of G (see [4]).

The main purpose of this paper is to perform a numerical verification of the asymptotic formula (9), working with the standard family $\tilde{F}_{\alpha,\varepsilon}$. The rotation numbers we select for the computations are quadratic irrationals, mainly focusing in the case when θ is the Golden Mean, $\theta = (\sqrt{5} - 1)/2$.

To compute the width of the strip of analyticity of the conjugation we are going to use a result due to Herman (see Proposition 3). Moreover, as

$$\Delta(\varepsilon) = \frac{1}{2\pi} \log R_\varepsilon = \tilde{\Delta}(\varepsilon) - \frac{1}{2\pi} \log \left(\frac{2}{\varepsilon} \right) \tag{13}$$

is an analytic (even) function of ε , we also adapt Herman's method to compute the Taylor expansion of $\Delta(\varepsilon)$ at $\varepsilon = 0$. Next to that, we compare this Taylor expansion with $\Delta(\varepsilon)$ computing this function for a table of values of $\varepsilon \in [0, 1]$.

Among the problems we have faced to perform these numerical computations, with enough precision to make a successful comparison with the Taylor expansion, here we want to stress two. First, the accurate computation of the Arnold Tongue T_θ . For this purpose, we have used a numerical method previously developed by the authors to compute rotation numbers with high precision (see [17], Section 4.2). Second, the improvement of the numerical results for $\Delta(\varepsilon)$ provided by Herman's method. To do that, we have combined the direct computation with some heuristic observations and semi-analytical ideas, in order to develop an *ad hoc* extrapolation process suited for the method, that depends strongly on the arithmetic properties of the selected rotation numbers.

To give a partial justification of these ideas, for the case of the Golden Mean, we have also adapted Herman's method for computing the Fourier coefficients of the periodic part of the conjugation (2).

We also mention that all the numerical computations have been implemented *ad hoc* in C++ code. Moreover, in order to perform the computations of the different quantities with enough precision to detect its asymptotic behavior, we have replaced the standard *double* data type of the computer by the so-called *double-double* data type, of approximately 32 decimal digits, which is provided by the *quad-double/double-double computational package* (see [12]).

The paper is structured as follows. In Section 2 we present Herman's result and show how it can be used to compute the function $\Delta(\varepsilon)$ as well as its derivatives. Moreover, in Section 2.3 we adapt this method for computing the Fourier coefficients of the conjugation. Section 3 is devoted to apply this methodology to the Arnold family. For the case of the Golden Mean, we develop an extrapolation method to improve Herman's method in Section 3.3. Moreover, we also give numerical evidences of the correctness of the asymptotic expansions used in this extrapolation process. In Section 3.4 we compute some Fourier coefficients of the conjugation and detect its asymptotic behavior. This behavior is used in Appendix B to give a partial justification of the asymptotics used in Section 3.3. In Section 3.5 we briefly discuss the case of other quadratic irrational rotation numbers. Finally, in Appendix A, we analytically compute some Taylor coefficients of the function $\alpha(\varepsilon)$ for any Diophantine rotation number θ . These coefficients are required in Section 3.2.

2. Computation of the size of the rotation domain

In Section 2.1 we introduce Herman's method to compute the size of a Siegel disk or a Herman ring of a map in the complex plane. Our next step is to translate this method in order to compute the size Δ of the rotation domain of a circle map. Later, in Section 2.2, we formulate this method in terms of a one-parameter family of circle maps f_μ , so that we can adapt it to the computation of the derivatives of the size $\Delta(\mu)$. Finally, in Section 2.3, a slight modification of the method is used to compute the Fourier coefficients of the conjugation.

2.1. Herman's method

Let F be an analytic map of \mathbb{C} , leaving \mathbf{C}_1 invariant, and f the induced map on \mathbb{T}^1 via the complex exponential; we suppose $f : \mathbb{T}^1 \rightarrow \mathbb{T}^1$ is an analytic circle diffeomorphism.

Remark 2. In what follows we are going to deal with a lift of $F|_{\mathbf{C}_1}$ to \mathbb{R} rather than the corresponding map on \mathbb{T}^1 . Thus, from now on we denote by f this lift, in the understanding that, to define the corresponding map on \mathbb{T}^1 , we only have to take modulo one in the formula of f . This construction is straightforward for the Arnold family.

We suppose that $F|_{\mathbf{C}_1}$ has a Diophantine rotation number θ , and we want to discuss how to compute the size R of its Herman ring and the size Δ of the rotation domain of f (see Remark 1). If we focus for instance on the definition of R , what we have to do, in principle, is to compute the Laurent expansion of the conjugation φ around \mathbf{C}_1 (see (11)). Then, we can obtain its outer radius of convergence from the behavior of the coefficients of this expansion. Of course, it is not realistic to expect that, by applying this method to $\tilde{F}_{\alpha(\varepsilon), \varepsilon}$ in (12), we can obtain a numerical approximation to \tilde{R}_ε with enough precision to detect its asymptotic behavior (9).

Alternatively, we proceed analogously as Marmi in [14], and use the following result due to Herman.

Proposition 3 (Herman, [11]). *Let F be an analytic map in a neighborhood of the origin, such that $F(0) = 0$ and $F'(0) = e^{2\pi i\theta}$. If φ linearizes F (see (8)) and we let $z = \varphi(w)$, with $|w| = r < R$, where R is the conformal radius of its Siegel disk U , we have that $z \in U$ and that*

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log |F^j(z)| = \log r. \tag{14}$$

Moreover, if $\{p_j/q_j\}_{j \geq 0}$ are the convergents of the continued fraction expansion of θ , then

$$\left| \frac{1}{q_j} \sum_{j=0}^{q_j-1} \log |F^j(z)| - \log r \right| \leq \frac{1}{q_j} \text{var}(\log |\varphi|_{\mathbf{C}_1}), \tag{15}$$

where $\text{var}(\cdot)$ is the variation of the curve.

This result can be generalized to the case of Herman rings of complex maps (see [14]). Moreover, if we suppose that we are able to take the limit when $r \rightarrow R^-$, then we can use (14) and (15) to compute R by taking $z \in \partial U$.

Let us explain what Proposition 3 means in terms of the circle map f and the size of its rotation domain Δ . We consider a point $a - i\Delta$ on the (lower) boundary of the strip of analyticity of the conjugation η in (2), with $a \in \mathbb{R}$, and we iterate $x^* = \eta(a - i\Delta)$ (assuming this point defined) by the action of f . By expanding ξ in Fourier series,

$$\xi(x) = \sum_{k \in \mathbb{Z}} \hat{\xi}_k e^{2\pi i k x}, \tag{16}$$

we obtain

$$f^n(x^*) = \eta(a - i\Delta + n\theta) = a - i\Delta + n\theta + \sum_{k \in \mathbb{Z}} \hat{\xi}_k e^{2\pi i k(a - i\Delta + n\theta)}.$$

Let us note that, ξ being a real analytic function, its Fourier coefficients verify $\hat{\xi}_{-k} = \hat{\xi}_k$, for any $k \in \mathbb{Z}$. In particular, $\hat{\xi}_0 \in \mathbb{R}$.

Now, if we denote by $\hat{\xi}_k = \hat{\xi}_k e^{2\pi i k(a - i\Delta)}$, for $k \neq 0$, $\hat{\xi}_0 = \hat{\xi}_0 + a - i\Delta$ and $\hat{f}_n = f^n(x^*) - n\theta$, we have

$$\hat{f}_n = \sum_{k \in \mathbb{Z}} \hat{\xi}_k e^{2\pi i k n \theta}. \tag{17}$$

In view of Proposition 3, we consider the sum of the first N iterates of the map

$$S_N = \sum_{n=0}^{N-1} \hat{f}_n = N \hat{\xi}_0 + \sum_{k \in \mathbb{Z} \setminus \{0\}} \hat{\xi}_k \frac{1 - e^{2\pi i k N \theta}}{1 - e^{2\pi i k \theta}}. \tag{18}$$

Hence, by assuming that the sum at the right-hand side of (18) divided by N goes to zero when $N \rightarrow +\infty$, we recover Herman's result¹

$$\lim_{N \rightarrow +\infty} \text{Im} \left(\frac{S_N}{N} \right) = -\Delta. \tag{19}$$

Remark 4. We point out that the fastest convergence speed we can expect for Δ in (19) is of $O(1/N)$, i. e., the same order of convergence expected when computing the rotation number of a circle map from its definition. Later, in Section 3.3, we are going to discuss how this convergence can be accelerated (for the Arnold family and θ being the Golden Mean) by means of a suitable extrapolation process.

The main difficulty in using (19) for computing Δ lies in knowing a point x^* on the boundary of the rotation domain of the circle map f . The most natural candidates are the critical points of the map, defined so that $f'(x^*) = 0$. It is clear that a critical point cannot be in the interior of any rotation domain, and a very important problem is to investigate if there is a critical point on its boundary. Herman showed in [11] that there are examples of maps without critical points on the boundary of their Siegel disk. However, there are several results in the positive direction (see for instance [6,7,9]). For our concerns, Geyer claimed in [5] (see [3] for a sketch of a proof) that the critical points of $\tilde{F}_{\alpha, \varepsilon}$ in (12) are always on the boundary of its Herman ring for rotation numbers θ of constant type² (the same also holds for the Siegel disk of the semistandard map G in (7)).

2.2. Variationals of Herman's method

Now we consider a parametric approach to formula (19). Let us suppose that $f_\mu : \mathbb{R} \rightarrow \mathbb{R}$ is a one-parameter family of real analytic maps, which are lifts of a one-parameter family of diffeomorphisms of the circle. We also suppose that the rotation number $\theta = \rho(f_\mu)$ is independent of μ and Diophantine. If the dependence on μ of the family f_μ is smooth enough (analytic in our context), one can ask if the function $\Delta(\mu)$ giving the size of the rotation domain of f_μ is also smooth. Assuming the answer positive, one can try to use formula (19) to compute the derivatives of $\Delta(\mu)$. For this purpose, we suppose known, for any μ , a (complex) point x_μ^* at the (lower) boundary of the rotation domain of f_μ . We also suppose that x_μ^* depends smoothly on μ (from the practical point of view x_μ^* has to be a critical point of the map f_μ). Supposing that formula (19) holds on the boundary, we have

$$\lim_{N \rightarrow +\infty} \text{Im} \left(\frac{1}{N} \sum_{n=0}^{N-1} (f_\mu^n(x_\mu^*) - n\theta) \right) = -\Delta(\mu).$$

Then, by taking derivatives with respect to μ , we obtain the following (formal) expressions

$$\lim_{N \rightarrow +\infty} \text{Im} \left(\frac{1}{N} \sum_{n=0}^{N-1} \frac{d^k}{d\mu^k} (f_\mu^n(x_\mu^*) - n\theta) \right) = -\Delta^{(k)}(\mu), \quad k \geq 0, \tag{20}$$

where the derivatives of $f_\mu^n(x_\mu^*)$ can be computed recurrently (see Section 3.2).

¹ Of course, this property is straightforward if instead of Δ we take any number y with $-\Delta < y < \Delta$.

² $\theta \in \mathbb{R} \setminus \mathbb{Q}$ is of constant type if its continued fraction expansion, $\theta = [a_0; a_1, a_2, \dots]$, verifies $a_0 \in \mathbb{Z}$, $a_n \in \mathbb{N}$ and $\sup_n a_n < +\infty$.

2.3. Fourier coefficients of the conjugation

Sometimes it is useful to compute the Fourier coefficients of $\xi(x)$ in (16). See for instance Section 3.4 for the case of the Arnold family.

For this purpose, we focus on formula (17) and on the modified Fourier coefficients $\hat{\xi}_k$, which are those of the Fourier expansion of ξ at the lower boundary of its domain of analyticity. The most natural method to compute them numerically is to truncate (17), and to consider the approximate linear relation thus obtained

$$\hat{f}_n \approx \sum_{|k| \leq K} \hat{\xi}_k e^{2\pi i k n \theta},$$

for certain $K > 0$. Thus, by computing a finite number of iterates of the map, we can obtain numerical approximations for $\hat{\xi}_k$, $|k| \leq K$, by solving a linear system of equations. We observe that the matrix of this system is Vandermonde-like, and the determinant is given by the product of quantities of the form $e^{2\pi i k \theta} - e^{2\pi i k' \theta} = e^{2\pi i k \theta} (1 - e^{2\pi i (k' - k) \theta})$. This means that the determinant is obtained from a product of “small divisors”, which can lead to an ill-conditioned system of equations.

In this section we discuss an alternative method, based on a modification of the definition of S_N in (18), allowing to compute the Fourier coefficients in the same way as Δ from (19). Given a fixed $k^* \in \mathbb{Z}$, we denote by $\hat{f}_n^{k^*} = \hat{f}_n e^{-2\pi i k^* n \theta}$. Hence, from (17) we have

$$\hat{f}_n^{k^*} = \sum_{k \in \mathbb{Z}} \hat{\xi}_k e^{2\pi i (k - k^*) n \theta}.$$

In this case, the sum of these modified iterates gives

$$S_N^{k^*} = \sum_{n=0}^{N-1} \hat{f}_n^{k^*} = N \hat{\xi}_{k^*} + \sum_{k \in \mathbb{Z} \setminus \{k^*\}} \hat{\xi}_k \frac{1 - e^{2\pi i (k - k^*) N \theta}}{1 - e^{2\pi i (k - k^*) \theta}}. \quad (21)$$

Then, under the same assumptions on the limit we made in (19), we obtain

$$\lim_{N \rightarrow +\infty} \frac{S_N^{k^*}}{N} = \hat{\xi}_{k^*}. \quad (22)$$

Remark 5. If f_μ is the one-parameter family of maps of Section 2.2, then we can (formally) compute the derivatives of $\hat{\xi}_{k^*}(\mu)$ by differentiating formula (22) analogously as we did in (20).

Remark 6. The direct evaluation of $e^{2\pi i k \theta} = \cos(2\pi k \theta) + i \sin(2\pi k \theta)$, for $k \geq 1$, is very expensive from the numerical point of view, but we observe that $\cos(2\pi k \theta)$ and $\sin(2\pi k \theta)$ can be computed recursively by using a recurrence that is numerically stable. Thus, we only need to compute $\cos(2\pi \theta)$ and $\sin(2\pi \theta)$.

Formula (22) provides the coefficient $\hat{\xi}_{k^*}$ for any $k^* \in \mathbb{N}$ (coefficients with $k < 0$ can be easily obtained from those with $k > 0$ and are exponentially small in $|k|$, of $O(e^{-4\pi \Delta |k|})$). However, we observe that (22) converges slowly as we increase k^* . The reason is that, for k^* big there are many coefficients $\hat{\xi}_k$ with $0 \leq k \ll k^*$ which have bigger size than $\hat{\xi}_{k^*}$. In the general case, one can use the following trick to overcome this problem. First, we use (22) to compute $\hat{\xi}_k$ for $0 \leq k \leq K$, with K not too big. From the numerical approximations thus obtained, namely $\{\hat{\xi}_k\}_{0 \leq k \leq K}$, we consider again formula (22), but now applied to

$$\bar{f}_n^{k^*} = \hat{f}_n^{k^*} - \sum_{0 \leq k \leq K} \hat{\xi}_k e^{2\pi i (k - k^*) n \theta}.$$

This new expression can be used to improve the numerical approximations $\{\hat{\xi}_k\}_{0 \leq k \leq K}$ or to compute new coefficients with

$k > K$. Of course, this process can be iterated but, unfortunately, this is more expensive than the direct method (22).

Nevertheless, in this paper we use another approach in order to improve $\hat{\xi}_k$, that takes advantage of the particular case we are considering. In Section 3.4, we apply formula (22) to compute these Fourier coefficients for the Arnold family (1), when the rotation number is the Golden Mean. Then, the experimental study of these coefficients gives us the chance to apply an extrapolation process to refine them.

3. Application to the Arnold family

In this section we consider the methods of Section 2 for the case when the map F is $\tilde{F}_{\alpha(\varepsilon), \varepsilon}$ in (12), and thus $f = \tilde{f}_{\alpha(\varepsilon), \varepsilon}$ in (1), where $\alpha = \alpha(\varepsilon)$ is the parameterization of an Arnold Tongue T_θ for the Arnold family, for a fixed Diophantine number θ . In the numerical experiments we display along this section we take θ to be the Golden Mean, except for in Section 3.5 where we explore the case of other quadratic irrational rotation numbers.

The map $\tilde{F}_{\alpha, \varepsilon}$ has two critical points located at

$$z_\pm^* = \frac{1}{\varepsilon} (-1 \pm \sqrt{1 - \varepsilon^2}) < 0.$$

If we use the transformation $z = e^{2\pi i x}$, we obtain the critical points of $\tilde{f}_{\alpha, \varepsilon}$:

$$x_\pm^*(\varepsilon) = \frac{1}{2} - \frac{i}{2\pi} \log \left(\frac{1 \pm \sqrt{1 - \varepsilon^2}}{\varepsilon} \right).$$

As we are interested in the critical point on the lower boundary, we pick up $x^* = x_+^*(\varepsilon)$.

3.1. Scaling the Arnold family

The first problem we face when trying to compute the asymptotic size of the rotation domain of $\tilde{f}_{\alpha(\varepsilon), \varepsilon}$ is that the function $\tilde{\Delta}(\varepsilon)$ in (5) is not bounded when $\varepsilon \rightarrow 0$. Nevertheless, as we know *a priori* that $\Delta(\varepsilon)$ in (13) can be analytically continued to \mathbb{D}_1 (see (9)), we perform a scaling on the Arnold family to focus on the computation of $\Delta(\varepsilon)$.

Thus, we introduce the change of variables $x = t - (i/2\pi) \log(2/\varepsilon)$ and denote by $f_{\alpha, \varepsilon}$ the Arnold family $\tilde{f}_{\alpha, \varepsilon}$ expressed in this new variable,

$$f_{\alpha, \varepsilon}(t) = t + \frac{\alpha}{2\pi} - \frac{i}{2\pi} e^{2\pi i t} + \varepsilon^2 \frac{i}{8\pi} e^{-2\pi i t}. \quad (23)$$

The map $f_{\alpha(\varepsilon), \varepsilon}$ is analytically conjugate to the rotation \mathcal{T}_θ through the (scaled) conjugation

$$\eta_\varepsilon(t) = \tilde{\eta}_\varepsilon \left(t - \frac{i}{2\pi} \log \left(\frac{2}{\varepsilon} \right) \right) + \frac{i}{2\pi} \log \left(\frac{2}{\varepsilon} \right), \quad (24)$$

defined in the (maximal) complex strip (see (4))

$$\mathbf{A}(\varepsilon) = \left\{ t \in \mathbb{C} : -\Delta(\varepsilon) < \text{Im}(t) < \Delta(\varepsilon) + \frac{1}{\pi} \log \left(\frac{2}{\varepsilon} \right) \right\}.$$

So, we apply Herman's method to compute the lower border of this strip,

$$\lim_{N \rightarrow +\infty} \text{Im} \left(\frac{1}{N} \sum_{n=0}^{N-1} (f_{\alpha(\varepsilon), \varepsilon}^n(t_\varepsilon^*) - n\theta) \right) = -\Delta(\varepsilon), \quad (25)$$

where

$$t_\varepsilon^* = \frac{1}{2} - \frac{i}{2\pi} \log \left(\frac{1 + \sqrt{1 - \varepsilon^2}}{2} \right) \quad (26)$$

is now the lower critical point of $f_{\alpha(\varepsilon), \varepsilon}$.

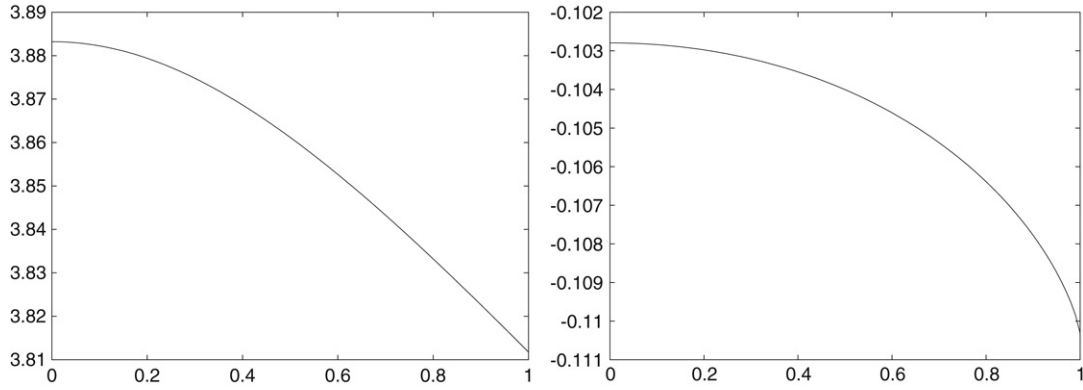


Fig. 1. Left: the graph of $\alpha(\varepsilon)$ for the Arnold Tongue for the Golden Mean. Right: the graph of $\Delta(\varepsilon)$ using (25).

In Fig. 1 it is plotted the function $\Delta(\varepsilon)$ obtained for the case of the Golden mean using $N = F_{34}$ iterates of the map, where $F_{34} = 9227\ 465$ is a Fibonacci number (see Section 3.3 for the motivation). We remark that, to perform these computations, we need to know the function $\alpha(\varepsilon)$, giving the Arnold Tongue, with enough precision. This precision is important to avoid big effects of the propagation of the error after doing a big number of iterates of the map. The function $\alpha(\varepsilon)$ has been obtained using a method introduced in [17] for computing the rotation number of a circle map with high precision (see Section 3.3 for a brief explanation of the method) and the secant method. See also [13] for a similar approach using the Newton method. Using [17], the function $\alpha(\varepsilon)$ has been computed (numerically) so that the rotation number of the points on “the tongue” is the Golden Mean with an (estimated) error smaller than 10^{-32} . The graph of $\alpha(\varepsilon)$ is also plotted in Fig. 1.

3.2. Explicit recurrences for the scaled map

Our purpose now is to apply the method of Section 2 to compute the Taylor expansion of $\Delta(\varepsilon)$. As all the quantities we are going to consider turn to be even with respect to ε , we introduce a new parameter $\mu = \varepsilon^2$. Abusing notation, in the rest of this section we are going to write $\Delta(\mu)$ instead of $\Delta(\varepsilon)$ and the same for the other ε -depending quantities. In this way, we introduce

$$f_\mu(t) = f_{\alpha(\varepsilon), \varepsilon}(t) = t + \frac{\alpha(\mu)}{2\pi} - \frac{i}{2\pi} e^{2\pi i t} + \mu \frac{i}{8\pi} e^{-2\pi i t}, \quad (27)$$

whose lower critical point is

$$t_\mu^* = \frac{1}{2} - \frac{i}{2\pi} \log \left(\frac{1 + \sqrt{1 - \mu}}{2} \right). \quad (28)$$

We focus on the first three coefficients of the Taylor expansion of $\Delta(\mu)$

$$\Delta(\mu) = \delta_0 + \mu \delta_1 + \mu^2 \delta_2 + \dots, \quad (29)$$

where $\delta_k = \Delta^{(k)}(0)/k!$.

The computation of δ_0 follows by applying (19) to the *semistandard map in the circle*, which is obtained through the identification $G(e^{2\pi i t}) = e^{2\pi i g(t)}$ (see (7)). It is given by the expression

$$g(t) = f_0(t) = t + \theta - \frac{i}{2\pi} e^{2\pi i t} \quad (30)$$

(recall $\alpha(0) = 2\pi\theta$) and has the critical point $t_0^* = 1/2$. Then, we have

$$\delta_0 = \lim_{N \rightarrow +\infty} \text{Im} \left(\frac{S_N}{N} \right) = \lim_{N \rightarrow +\infty} \text{Im} \left(\frac{1}{N} \sum_{n=0}^{N-1} \hat{g}_n \right), \quad (31)$$

where $\hat{g}_n = g^n(1/2) - n\theta$.

To compute $\Delta^{(k)}(0)$ for $k = 1, 2$, we use (20). So the main point is to obtain the derivatives of the iterates of the critical point, $\frac{d^k}{d\mu^k} (f_\mu^n(t_\mu^*))|_{\mu=0}$, which can be computed recursively. More precisely, we introduce

$$u_n(\mu) = f_\mu^n(t_\mu^*), \quad v_n(\mu) = u_n'(\mu), \quad w_n(\mu) = u_n''(\mu),$$

and then from (27) we obtain the following recurrences:

$$\begin{aligned} u_{n+1}(\mu) &= f_\mu(u_n(\mu)) \\ &= u_n(\mu) + \frac{\alpha(\mu)}{2\pi} - \frac{i}{2\pi} e^{2\pi i u_n(\mu)} + \mu \frac{i}{8\pi} e^{-2\pi i u_n(\mu)}, \\ v_{n+1}(\mu) &= v_n(\mu) + \frac{\alpha'(\mu)}{2\pi} + e^{2\pi i u_n(\mu)} v_n(\mu) \\ &\quad + \frac{i}{8\pi} e^{-2\pi i u_n(\mu)} + \frac{\mu}{4} e^{-2\pi i u_n(\mu)} v_n(\mu), \\ w_{n+1}(\mu) &= w_n(\mu) + \frac{\alpha''(\mu)}{2\pi} + 2\pi i e^{2\pi i u_n(\mu)} v_n^2(\mu) \\ &\quad + e^{2\pi i u_n(\mu)} w_n(\mu) + \frac{1}{2} e^{-2\pi i u_n(\mu)} v_n(\mu) \\ &\quad - \mu \frac{\pi i}{2} e^{-2\pi i u_n(\mu)} v_n^2(\mu) + \frac{\mu}{4} e^{-2\pi i u_n(\mu)} w_n(\mu). \end{aligned}$$

In particular, if we set $\mu = 0$, we have

$$\begin{aligned} u_{n+1}(0) &= u_n(0) + \theta - \frac{i}{2\pi} e^{2\pi i u_n(0)}, \\ v_{n+1}(0) &= v_n(0) + \frac{\alpha'(0)}{2\pi} + e^{2\pi i u_n(0)} v_n(0) + \frac{i}{8\pi} e^{-2\pi i u_n(0)}, \\ w_{n+1}(0) &= w_n(0) + \frac{\alpha''(0)}{2\pi} + e^{2\pi i u_n(0)} (2\pi i v_n^2(0) + w_n(0)) \\ &\quad + \frac{1}{2} e^{-2\pi i u_n(0)} v_n(0). \end{aligned} \quad (32)$$

By expanding (28) in power series we obtain the seeds of this iterative process,

$$u_0(0) = \frac{1}{2}, \quad v_0(0) = \frac{i}{8\pi}, \quad w_0(0) = \frac{3i}{32\pi}.$$

The only remaining question to apply recurrences (32) is to compute the Taylor expansion of the Arnold Tongue $\alpha = \alpha(\mu)$. In Appendix A we analytically compute the first terms of this expansion, obtaining

$$\begin{aligned} \alpha(0) &= 2\pi\theta, \quad \alpha'(0) = \frac{\cos \pi\theta}{4 \sin \pi\theta}, \\ \alpha''(0) &= -\frac{3 + \cos 4\pi\theta}{64(\sin \pi\theta)^2 \sin 2\pi\theta}. \end{aligned}$$

