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Numerical computation of rotation numbers of quasi-periodic planar curves

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ABSTRACT

Recently, a new numerical method has been proposed to compute rotation numbers of analytic circle diffeomorphisms, as well as derivatives with respect to parameters, that takes advantage of the existence of an analytic conjugation to a rigid rotation. This method can be directly applied to the study of invariant curves of planar twist maps by simply projecting the iterates of the curve onto a circle. In this work we extend the methodology to deal with general planar maps. Our approach consists in computing suitable averages of the iterates of the map that allow us to obtain a new curve for which the direct projection onto a circle is well posed. Furthermore, since our construction does not use the invariance of the quasi-periodic curve under the map, it can be applied to more general contexts. We illustrate the method with several examples.

1. Introduction

In this paper we present numerical algorithms to deal with quasi-periodic invariant curves of planar maps by adapting a method presented in [1] to compute rotation numbers of analytic circle diffeomorphisms. The developed ideas do not require the curve to be invariant under any map; so they can be applied to more general objects that we refer to as *quasi-periodic signals* (see Definition 2.2).

The method of [1] is built assuming that the circle map is analytically[1] conjugate to a rigid rotation and, basically, it consists in computing suitable averages of the iterates of the map followed by the Richardson extrapolation. Since this construction takes advantage of the geometry and the dynamics of the problem, the method turns out to be highly accurate and very efficient in multiple applications. In a few words, if we compute N iterates of the map, then we can approximate the rotation number with an error of order $\mathcal{O}(1/N^{p+1})$ where p is the selected order of averaging (compared with $\mathcal{O}(1/N)$ obtained using the definition). This methodology has been extended in [2] to deal with derivatives of the rotation number with respect to parameters. In this case, it is required to compute and average the corresponding derivatives of the iterates of the circle map. We want to point out that this variational information cannot be obtained in such a direct way by means of other existing methods to compute rotation numbers (we refer to the works [3–8]).

As a matter of motivation, let us assume first that F is a map on the real annulus $\mathbb{T} \times I$, where I is a real interval and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and let $X : \mathbb{T} \times I \to \mathbb{R}$ denote the canonical projection $X(x, y) = x$. If F is a twist[2] map, the Birkhoff Graph Theorem (see [9]) ensures that every invariant curve Γ is a graph over its projection on the circle by means of X, and its dynamics induces a circle map by projecting the iterates. Hence, it is straightforward to apply the method of [1] in order to approximate the rotation number of Γ, since for any $(x_0, y_0) \in \Gamma$ we can compute the orbit $x_n = X(F^n(x_0, y_0))$ — this is the only data that the method requires. Furthermore, if F has a differentiable family of invariant curves or a Cantorian family differentiable in the sense of Whitney, we can approximate derivatives of the rotation number with respect to initial conditions and parameters. This allows us to implement a Newton scheme for the computation and continuation of invariant curves of twist maps (as it is discussed in detail in [2]).

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[1] The methods of [2,1] also work in the class of $\mathcal{C}^r$ circle diffeomorphisms, r being sufficiently large, but we restrict the discussion to the analytic case in order to simplify the exposition.

[2] The map F satisfies the *twist condition* if $\partial(X \circ F)/\partial y$ does not vanish.

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If the map does not satisfy the twist condition or it is not written in suitable coordinates, its invariant curves are not necessarily graphs over the projection on a circle. In this situation, invariant curves can fold in a very wild way (see Section 3.3 and references given therein for examples of such curves). Nevertheless, if we can select a suitable circle so that the folded curve "rotates" around it, then the projection of the iterates of the map does not define a circle map but a "circle correspondence" and we can compute the rotation number of the curve from the "lift" of this correspondence to the real line — see Section 2.1 for details. Moreover, albeit we do not have a justification of this fact, we realize that the extrapolation methods of [2,1] work quite well when applied to the iterates of this "lift".

In some cases – for example, if the rotation number is large compared with the size of the folds – we can compute numerically this "lift" from the iterates of the map. However, if the curve is extremely folded additional work is required in order to face the problem in a systematic way. Hence, we propose a numerical method to construct a circle map – preserving the rotation number – from a general invariant curve on the plane. The method consists in averaging the iterates of an orbit of the curve in such a way that the new iterates belong to another curve, no longer invariant under the map, but having the same rotation number. Concretely, if we know an approximation of the rotation number with error ε, we construct a sequence of (averaged) curves that approaches a circle up to terms of order $\mathcal{O}(\varepsilon)$. We refer to this construction as *unfolding of the curve* since if ε is small enough, then this construction provides us with a circle map. Taking into account the discussion in the previous paragraph, in order to apply the methods of [2,1] it is not necessary to unfold completely the curve, but only to "kill" its main folds so we can compute the "lift" of the correspondence generated by the projection of the iterates of the new (less-folded) curve. In order to justify this unfolding procedure we require the curve to be analytic (or at least differentiable enough) and the rotation number to be Diophantine. Sometimes the requested approximation of the rotation number is given by the context of the problem – for example, if we look for invariant curves of fixed rotation number – or it can be obtained by means of any method of frequency analysis (see for example [6,7]). Therefore, we obtain a very efficient toolkit for the study of invariant curves of planar maps and their numerical continuation.

Let us remark that due to the importance, both theoretical and applied, of invariant curves of maps or two-dimensional tori of flows (for example, they play a fundamental role in the design of space missions [10,11] and also in the study of models in Celestial Mechanics [12], Molecular Dynamics [13,14] or Plasma-Beam Physics [15], just to say a few), several approaches to deal with these objects have been developed in the literature. For example, the methods in [16–18] have been applied efficiently in a wide set of contexts. However, they require to compute a representation – by means of a trigonometric polynomial – of the curve which solves the invariance equation of the problem, so it is required to solve large systems of equations — as large as the used number of Fourier modes, say M. One possibility to face this difficulty is to solve these full linear systems, with a cost $\mathcal{O}(M^3)$ in time and $\mathcal{O}(M^2)$ in memory, by means of efficient parallel algorithms as is proposed in [18]. Another recent approach presented in [17], based on the analytic and geometric ideas developed in [19], allows us to reduce the computational effort of the problem to a cost of order $\mathcal{O}(M \log M)$ in time and $\mathcal{O}(M)$ in memory. On the other hand, we can compute the invariant curve by looking for a point so that the corresponding orbit has a prefixed rotation number. Then, rather than approximating explicitly the parameterization of the curve, we reduce the problem to finding a zero of a function. This approach can be implemented using interpolation methods as in [20] or also using the extrapolation methods in [2,1]. These extrapolation methods, that are the cornerstone of the presented paper, have a cost of order $\mathcal{O}(N \log N)$ in terms of the used number of iterates N and are free in memory. Once we know a point on the curve and its rotation number, we can compute a trigonometric approximation of the curve "a posteriori", using Fourier Transform (FT) on the iterates of the curve. In addition, in Section 2.7 we develop a method for performing this FT based also on averaging-extrapolation ideas.

Given a numerical method for the continuation of invariant curves, it is specially interesting to verify if the method is valid up to the *breakdown threshold* corresponding to the *critical invariant curve* (see [21–23]). These critical curves are specially important objects that organize the long-term behavior of a given dynamical system, because of their role as "last barriers" or "bottlenecks" to chaos (see [9]). Actually, the critical value for the breakdown of the golden curve for the Chirikov standard map was estimated by means of extrapolation methods in [1] obtaining a good agreement with the value predicted by means of the classical Greene criterion in [22]. For the non-twist case, we refer to computations in [24,25] as examples of breakdown studies in non-twist maps. It is worth mentioning that the methods presented in this paper can be applied also in this context.

Since our construction does not use the invariance of the curve under the map, it can be applied to the study of quasi-periodic curves that are not necessarily embedded (that we call quasi-periodic signals). This context is very interesting since it allows us to analyze sets of data obtained from real experiments or observed natural phenomena. Actually, in order to check that the methods are robust when facing experimental data, we consider the effect of Gaussian error in the evaluation of iterates of a known quasi-periodic function.

We want to point out that our approach can be also understood as a method for the refinement of the frequency analysis of [7]. Actually, an efficient refinement of these methods, based in the simultaneous improvement of the frequencies and the amplitudes of the signal, is given in [6]. Once again, the main advantage of our approach is that we do not have to compute Fourier coefficients of the curve. This fact reduces the computational effort of solving big linear systems of equations required to refine the representation of the signal. In addition, the accuracy in the computation of the rotation number is not limited by the truncation error in the representation of the signal.

Finally, we notice that the methodology of [2,1] also works for dealing with maps of the d-dimensional torus that admit an analytic conjugation to a rigid rotation having a Diophantine rotation vector. Our aim is to explore the extension of the ideas presented in this paper to deal with invariant tori and quasi-periodic signals of arbitrary number of frequencies.

The paper is organized as follows. In the first part, contained in Section 2, we develop and justify different results, methods and algorithms to study quasi-periodic invariant curves (or quasi-periodic signals). In the second part, presented in Section 3, we consider several examples in order to illustrate different features of the presented methodology. These examples have been selected in order to sustain the presentation of the methods and to highlight both some of the possibilities and limitations of our approach.

2. Exposition of methods

As we said in the introduction, we approach the study of quasi-periodic signals by computing the rotation number of a circle map (or a circle correspondence) induced by the curve. The main definitions and notation, together with a brief overview of the problem, are given in Section 2.1. After that, we present and justify a method to unfold a quasi-periodic signal. We first assume in Section 2.2 that the rotation number is known exactly in order to

Fig. 1. Left: Folded invariant curve with quasi-periodic dynamics that rotates around the origin in the complex plane (this curve corresponds to an example discussed in Section 3.3). Right: "Lift" of the associated circle correspondence given by (1).

highlight the involved ideas. Basically, we construct a sequence of curves that converges to a circle whose dynamics corresponds to a rigid rotation. In Section 2.3 we assume that we only have an approximation of the rotation number and we show that the previous construction allows us to obtain a curve that is $\mathcal{C}^1$-close to be a circle — the proximity being of the same order as the error in the initial guess of the rotation number.

In order to obtain the required approximation, a possibility is to resort to frequency analysis methods. In Section 2.4 we review Laskar's frequency analysis method in terms of the language presented in this paper, just to stress that the same algorithms derived to unfold the curve can be adapted to obtain the required approximation of the rotation number as alternative of the classical methods.

For the sake of completeness we include in Section 2.5 a brief survey of the methods of [2,1] to compute the rotation number and derivatives with respect to parameters of the obtained circle map or correspondence. This review is necessary to understand the higher order method that we develop in Section 2.6 to improve the unfolding of curves. During the exposition it will be clear that the ideas used in the unfolding are related to FT. This fact is exploited in Section 2.7 in order to extrapolate Fourier coefficients once the rotation number is known.

2.1. Setting of the problem

For convenience, we identify the real plane with the set of complex numbers by defining $z = u + \mathrm{i}v$ for any $(u, v) \in \mathbb{R}^2$. Let $\Gamma \subset \mathbb{C}$ be a quasi-periodic invariant curve for a map $F : U \subset \mathbb{C} \to \mathbb{C}$ of rotation number $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Let us assume, for example, that the curve "rotates" around the origin and that it is a graph of the angular variable. Then, the projection

$$\begin{array}{ccc} \Gamma & \longrightarrow & \mathbb{T} \\ z & \longmapsto & x = \arg(z)/2\pi \end{array} \tag{1}$$

generates a circle map induced by the dynamics of $F|_\Gamma$. On the other hand, if Γ is folded, then the projection (1) does not provide a circle map, but defines a correspondence on $\mathbb{T}$ that we can "lift" to $\mathbb{R}$. For example, in the left plot of Fig. 1 we show a "folded" invariant curve on the complex plane for an example considered in Section 3.3. In the right plot of Fig. 1 we show the "lift" of the correspondence on $\mathbb{T}$ given by (1). Since the rotation number of the curve is no more than the averaged number of revolutions per iterate, it is not surprising that we can compute it as $\lim_{n\to\infty}(x_n - x_0)/n$, where x_n are the iterates under the "lift" to $\mathbb{R}$ of the circle correspondence. In this situation, we have observed that the methods of [2,1] can be applied to such a "lift" (see examples in Section 3.3), even though we do not have a justification of this fact.

In some cases, for example if the rotation number is large enough as to avoid the folds, we can compute numerically the "lift" of (1) using the iterates of an orbit. However, if the invariant curve presents large folds or we cannot identify directly a good point around which the curve is rotating, we cannot compute this "lift" in a systematic way. Then, our aim is to construct another curve, having the same rotation number, by using suitable averages of iterates of the original map. If we manage to eliminate (or at least minimize) the folds in the new curve, then we are able to obtain a circle diffeomorphism (or at least a circle correspondence that we can "lift" numerically).

As Γ is a quasi-periodic invariant curve of rotation number θ, there exists an analytic embedding $\gamma : \mathbb{T} \to \mathbb{C}$ verifying $\Gamma = \gamma(\mathbb{T})$ and

$$F(\gamma(x)) = \gamma(x + \theta).$$

In this situation, since the parameterization γ is periodic, we can use the Fourier series

$$\gamma(x) = \sum_{k\in\mathbb{Z}} \hat{\gamma}_k \mathrm{e}^{2\pi \mathrm{i}kx},$$

and, moreover, for a given $z_0 \in \Gamma$ we can ask for $\gamma(0) = z_0$. Then, the iterates of z_0 under F can be expressed using γ as

$$\begin{aligned} z_n &= F^n(z_0) = F^n(\gamma(0)) = F^{n-1}(\gamma(\theta)) \\ &= \gamma(n\theta) = \sum_{k\in\mathbb{Z}} \hat{\gamma}_k \mathrm{e}^{2\pi \mathrm{i}kn\theta}. \end{aligned} \tag{2}$$

As we will see, our method does not use the invariance of Γ under F but only the expression (2) for the iterates. Furthermore, even if we start with an invariant curve of a map, the intermediate stages of our construction may produce curves that are not embedded in $\mathbb{C}$. Using this fact as a motivation, we state the following definitions:

Definition 2.1. We say that a complex sequence $\{z_n\}_{n\in\mathbb{Z}}$ is a *quasi-periodic signal* of rotation number θ if there exists a periodic function $\gamma : \mathbb{T} \to \mathbb{C}$ such that $z_n = \gamma(n\theta)$. We also call $\Gamma = \gamma(\mathbb{T})$ a quasi-periodic curve.

Definition 2.2. Under the above conditions, let $\{z_n\}_{n\in\mathbb{Z}}$ be a quasi-periodic signal. Then, for any $\theta_0 \in \mathbb{R}$ and $L \in \mathbb{N}$, we define the following iterates

$$z_n^{(L,\theta_0)} = \frac{1}{L} \sum_{m=n}^{L+n-1} z_m \mathrm{e}^{2\pi \mathrm{i}(n-m)\theta_0}. \tag{3}$$